Sample Problems on the Invariance Principle

Irish Mathematical Olympiad Training, 2013

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Problem 1:

Let n be an odd positive integer. A teacher writes the numbers $1, 2, 3, \ldots, 2n$ on the blackboard. He then picks out two numbers a and b, erases them, and writes the number $|a - b|$ instead.

Prove that if he continues this process, an odd number will remain at the end.

Problem 2:

There are 13 white chameleons, 15 black chameleons, and 17 red chameleons on an island. When any two chameleons of different colours meet, they both change into the third colour.

Is it possible for all chameleons to eventually have the same colour?

Problem 3:

The number 2^{2013} is replaced by its digital sum (i.e., the sum of its digits). Then this number is replaced by its digital sum, and we continue the process.

If at some stage a 10-digit number remains, prove that this number must have a repeated digit.

Problem 4:

A circle is divided into 6 sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written, clockwise, into the sectors. You may now perform a series of steps whereby at each step, two neighbouring numbers are increased by 1.

Is it possible to end up with all the numbers equal?

Problem 5:

Find all nonnegative integers x, y, z such that $x^3 + 2y^3 = 4z^3$.

Problem 6:

Prove that $\sqrt{2}$ is irrational.

Problem 7:

$2n$ points are given in the plane, no three of which are collinear. Exactly n of these points are farms: call them $F_1, F_2, \ldots F_n$. The remaining n points are wells: $W_1, W_2, \ldots W_n$. It is intended to build a straight line road from each farm to a well, so that no well serves more than one farm. Show that it is possible to do this in a way such that none of the roads intersect.

Problem 8:

Starting with a 4-tuple $S = (a, b, c, d)$ of positive integers, we derive the transformed sequence $T(S) = (|a - b|, |b - c|, |c - d|, |d - a|)$. Does the sequence $S_1 = S, S_2 = T(S_1), S_3 = T(S_2), \ldots$ always end up at $(0, 0, 0, 0)$?

Problem 9:

47 points are given in the plane, no three of which are collinear. Divide these into 5 groups, with a *different* number of points in each group. Let N denote the number of triangles with vertices in *different* groups.

How should the points be divided in order to maximize N?