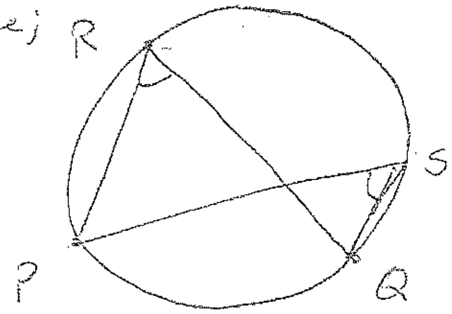
1.

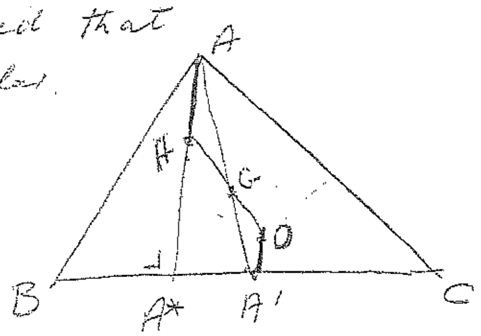
Centres of Triangles 22nd March

Here are the notes which we didn't get to, or did quickly.
We will continue the same notation:
[AA'], [BB'] and [CC'] denote the medians of $\triangle ABC$,
[AA*], [BB*], [CC*] " " altitudes " ",
O is the circumcircle of " ",
G is the centroid of " ",
H is the orthocentre of " ".

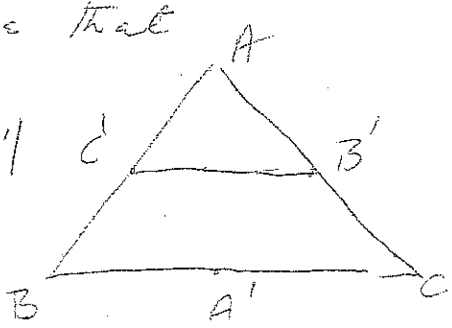
Recall:
(A) The theorem in J.C. that states that if PQ is an arc of a circle and R and S are points on the circle, but not on the arc, then $|\angle PRQ| = |\angle PSQ|$.
The converse is also true;
if $|\angle PRQ| = |\angle PSQ|$
then PRSQ is a
cyclic quadrilateral.



Recall:
(B) When we proved that H, G and O are collinear (on the Euler line) we showed that the $\triangle$s AHG and OGA' are similar.
We noted that $|AH| = 2|OA'|$,
which was irrelevant then
but we will need now.



Recall:
(C) The J.C. Theorem which states that the $\triangle$s AC'B' and ABC are similar, so $2|C'B'| = |BC| = 2|BA'|$ and $C'B' \parallel BC$.



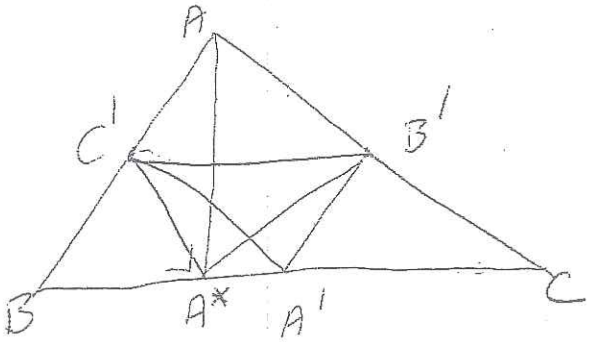
2. We use the notation we used in class.

(a) A^*, B^* and C^* lies on the circumcircle of $\triangle A'B'C'$ making a "six-point circle", sometimes called Feuerbach's circle,

(b) three other points, which we defined on the altitudes, lie on the same circle — "the nine-point circle" it is usually called,

c) the centre N of the nine-point circle lies on the Euler line.

(a) Our first step is to show that $A'B'C'A^*$ is a cyclic quadrilateral.



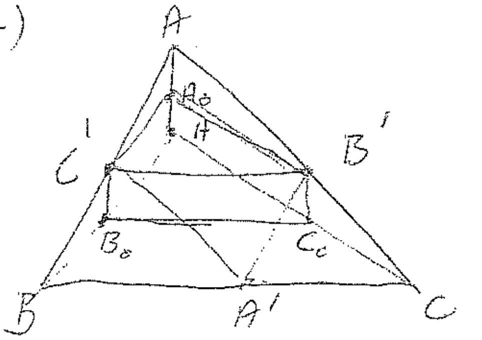
We know that
$\triangle AC'B'$ is congruent to $\triangle B'C'A'$ ($AB'A'C'$ is a parallelogram) and
$\triangle AC'B'$ is congruent to $\triangle A^*C'B'$ ($C'B'$ is the perpendicular bisector of $[AA^*]$).

So, $\triangle B'C'A'$ and $\triangle A^*C'B'$ are congruent, with $|\angle C'A^*B'| = |\angle C'A'B'|$. We conclude from Theorem (A) on page 1 that $A'B'C'A^*$ is a cyclic quadrilateral.

That is, A^* lies on the circumcircle of $\triangle A'B'C'$.

3. Similarly, we can show that B^* and C^* also lie on the circumcircle of $\triangle A'B'C'$.
Now, we have six-points of the nine-point circle.

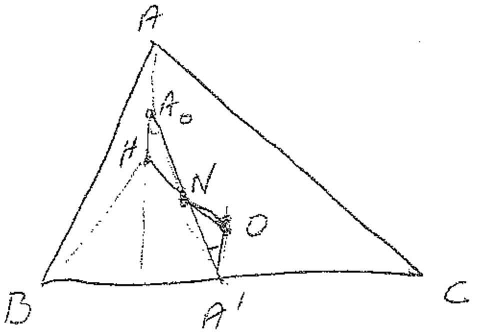
(b)



A_0 is the midpoint of $[AH]$,
B_0 " " " $[BH]$,
C_0 " " " $[CH]$.

We will show that A_0, B_0 and C_0 lie on the same circle, the nine-point circle.
Consider the $\triangle BAH$. Since $|AC'| = |C'B|$ and $|HB_0| = |B_0B|$, we can conclude from Theorem E on page 1 that $C'B^0 \parallel AH$.
Similarly, by considering the $\triangle CAH$, we have $B'C^0 \parallel AH$. Thus, $C'B^0 \parallel B'C^0$.
From $\triangle BHC$, we have $B^0C^0 \parallel BC$.
Note also that $AH \perp BC$ (AH is an altitude).
Therefore, $C'B'C_0B_0$ is a rectangle and so is cyclic with diameter $[C'C_0]$.
Similarly, we can show that $C'A_0 \parallel A'C_0$ and $C_0A' \parallel A_0C'$ so $C'A_0C_0A'$ is also a rectangle with diameter $[CC_0]$ for its circumcircle, i.e. the same diameter. So, those two rectangles have a common circumcircle. In other words, A_0, B_0 and C_0 lie on the circumcircle of $\triangle A_1B_1C_1$ which is the nine-point circle.
We'll call the centre of this circle N.

(c) We will prove that the "triangle centre" N that is the centre of the nine point circle, also lies on the Euler line.



From page 3, we know that $[C_0C']$, $[A_0A']$ and $[B_0B']$ are diameters of the nine-point circle. So N is the midpoint of, say, $[A_0A']$.

Consider the $\triangle A_0HN$ and the $\triangle A'ON$.

Now, $|A_0H| = |OA'|$, (see page 1, Recall (B))

$|A_0N| = |AN|$ and $|\angle HA_0N| = |\angle NA'O|$ ($AH \parallel OA'$)

So these $\triangle$s A_0HN and $A'ON$ are congruent.

Thus $|\angle HNA_0| = |\angle ONA'|$. We know that A_0, N and A' are collinear, so these are truly opposite angles. Therefore H, N and O must be collinear. That is, N lies on the Euler line.

Recall that $3|OG| = |OH|$.

We also have from the above that $|HN| = |NO|$, so, $2|ON| = |OH| = 3|OG|$. The line segments connecting the points of the Euler line that we've met, are separated by lengths in small natural number ratios to each other.

(5) Three other points on the Euler line you might like to look up (eg Wikipedia) are the de Longchamps point (1886), and (more recently defined) the Schiffler point (1985) and the Exeter point (1986).

Note that the incentre I lies on the Euler line only in an isosceles triangle. You could try to prove this.

I attach a short list of popular maths books that you might enjoy reading. The first one is the most accessible and the last one takes a bit of work (but is great if you persevere) — the other three are in between.

I hope you all continue to enjoy maths and come back for more "maths enrichment" next year if you're still in school.

Best wishes,

Mary Hanley

Popular maths books

Mary Hanley

The books that I've listed here can be enjoyed by people who have an interest in maths without having studied maths at third level.

- *Alex's Adventures in Numberland* (2010), by Alex Bellos,
- *Fermat's Last Theorem* (1997), by Simon Singh,
- *The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography* (2000), by Simon Singh,
- *The Music of the Primes* (2003), by Marcus du Sautoy,
- *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics* (2003), by John Derbyshire.