EGMO and UCD Enrichment Programme in Mathematics

Selection Test 16 February 2019: Solutions

1. The edges of a cube are coloured with three colours such that each vertex is the endpoint of an edge of each of the three colours. Show that there are four parallel edges of the same colour.

   Solution: Every face of the cube has four edges. Those four edges cannot be the same colour (as each vertex of the face has two edges of that face that must be different colours).

   Therefore, every face must be either*dichromatic*, ie the edges have two colours, or *trichromatic* (three different colours). For a dichromatic face, the opposite edges must be the same colour. For a trichromatic face, the two edges with the same colour must be opposite each other.

   We now consider two cases: (a) the cube has no trichromatic faces and (b) the cube has one or more trichromatic faces.

   If there are no trichromatic faces, then we are done, because a dichromatic face must have the same colour on opposite edges. Therefore any four parallel edges are the same colour.

   If there are trichromatic faces then arrange the cube so the upper face is trichromatic, and that the left and right edges of that face are both (without loss generality) green, while the front edge is red and the back is blue.

   Now the other edges are determined as follows:

   - The front left edge is blue (as upper left is green and upper front is red)
   - The back left edge is red (as upper left is green and upper back is blue)
   - Therefore the lower left edge is green, the same colour as the upper left and upper right edges.

   By the same argument reflect left and right, the lower right edge is green. Hence, the four parallel edges: upper left. upper right, lower left and lower right are all green and the result is proved.

2. Let $ABCD$ be a convex quadrilateral. Suppose that $AB = CD$. Prove that

   $$BC \cdot (\sin \angle B - \sin \angle C) = AD \cdot (\sin \angle A - \sin \angle D).$$

   Solution: We have

   $$\text{Area}(ABCD) = \text{Area}(ABC) + \text{Area}(ACD) = \text{Area}(ABD) + \text{Area}(BDC).$$

   This gives

   $$\frac{1}{2}AB \cdot BC \sin \angle B + \frac{1}{2}CD \cdot AD \sin \angle D = \frac{1}{2}AB \cdot AD \sin \angle A + \frac{1}{2}BC \cdot CD \sin \angle C$$

   and hence (multiplying by 2 and transposing terms)

   $$AB \cdot BC \sin \angle B - BC \cdot CD \sin \angle C = AB \cdot AD \sin \angle A - CD \cdot AD \sin \angle D.$$

   Now divide all terms by $AB = CD$.

3. Let a, b, c be the sides of a triangle. Prove that

   $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

   Solution: Assume without any loss of generality that c is the largest side of triangle. Using $c > a$, $c > b$ and $c < a + b$ we estimate

   $$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < \frac{a}{a+b} + \frac{b}{a+b} + \frac{c}{a+b} = 1 + \frac{c}{a+b} < 2.$$

Note. Here is another possible solution. Since the sum of any two sides is larger than the third side of the triangle, we estimate

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < \frac{a+a}{a+b+c} + \frac{b+b}{c+a+b} + \frac{c+c}{a+b+c} = 2.$$

4. Let $ABCD$ be a square. Let P be any point on the circumcircle of $ABCD$ lying on the arc joining A to B (and distinct from A and B). Let M be the point of intersection of DP with the diagonal AC. Let N be the point of intersection of CP with the side AB. Show that MN is parallel to BD.

   Solution: $\angle MAN = \angle CAB = 45°$. $\angle MPN = \angle DPC = \angle DBC$ (standing on same arc) $= 45°$. Thus $MAPN$ is a cyclic quadrilateral.

   Now $\angle BDP = \angle BAP$ (same arc) $= \angle NAP$. Since $MAPN$ is cyclic, $\angle NAP = \angle MNP$.

   Thus $\angle BDP = \angle MNP$ and hence the segment MN is parallel to the segment BD (they make the same angle with DP).

5. For any positive integer $n \geq 1$ denote $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. Prove that:

   (a) for any integer $n \geq 1$ we have

$$\frac{(2n)^2}{(2n-1)!(2n+1)!} < \frac{1}{(2n-1)!} - \frac{1}{(2n+1)!}.$$

   (b) We have

$$\frac{2^2}{1!3!} + \frac{4^2}{3!5!} + \frac{6^2}{5!7!} + \dots + \frac{2018^2}{2017!2019!} < 1 - \frac{1}{2019!}.$$

   Solution: (a) The right-hand-side of the inequality is

$$\frac{(2n+1)! - (2n-1)!}{(2n+1)!(2n-1)!}.$$

   The numerator here is

$$(2n+1)! - (2n-1)! = ((2n+1)\cdot 2n - 1)\cdot(2n-1)! = ((2n)^2 + (2n-1))\cdot(2n-1)! > (2n)^2 \text{ since } 2n-1 \geq 1.$$

   (b) Using (a) above we have

$$\frac{2^2}{1!3!} < \frac{1}{1!} - \frac{1}{3!}$$
$$\frac{4^2}{3!5!} < \frac{1}{3!} - \frac{1}{5!}$$
$$\frac{6^2}{5!7!} < \frac{1}{5!} - \frac{1}{7!}$$
$$\dots\dots\dots$$
$$\frac{2018^2}{2017!2019!} < \frac{1}{2017!} - \frac{1}{2019!}$$

   Adding the above inequalities we achieve the desired conclusion.

6. Finn has 5 distinct real numbers. He takes the sum of each pair of numbers and writes down the 10 sums. The 3 smallest sums are $30, 34$ and 35, while the 2 largest are 46 and 49.

   Determine, with proof, the largest of Finn's 5 numbers.

   Solution: Denote Finn's 5 numbers by $a < b < c < d < e$.

   Out of the 10 sums $30, 34, 35, \dots, 46, 49$, the largest is $d+e$ and the second largest is $c+e$. So $d+e = 49$ and $c+e = 46$.

   Furthermore, $a+b$ is the samllest sum and $a+c$ the second smallest. So $a+b = 30$ and $a+c = 34$.

The third smallest sum is either $a+d$ or $b+c$. However, we know that

$$a+d=(a+c)+(d+e)-(c+e)=34+49-46=37.$$

So $a+d$ is not the third smallest and hence $b+c=35$.

Thus, using the information we have so far we find that

$$2e=(c+e)+(d+e)-(a+d)-(b+c)+(a+b)=46+49-37-35+30=53,$$

and hence $e=53/2$.

7. Let ABC be a right-angled triangle with hypotenuse AB. Let D lie on the segment BC with $BD=2\cdot DC$. Let M be the midpoint of the hypotenuse AB. Determine the ratio AD/DM.

   Solution: Let A' lie on AC extended so that $AC=CA'$. Then ABA' is an isosceles triangle and BC is a median. It follows that D is the centroid of ABA'. Hence AD is a median of ABA' and it meets BA' at its midpoint, M' say.

   Since D is the centroid, $AD=2\cdot DM'$. But $DM'=DM$ by symmetry (triangles ABC and $A'BC$ are congruent).

   Conclusion: $AD/DM=2$.

8. Let S be a set of $6n$ points on a line. $4n$ of these points are painted blue and the other $2n$ points are painted green.

   Prove that there exists a line segment that contains exactly $3n$ points from S, such that $2n$ of them are blue and the other n are green.

   Solution: Let A_i be the line segment containing exactly $3n$ points from S, and whose left-hand endpoint is the ith point of S reading from left to right $(i=1,2,\ldots,3n+1)$. Let $f(i)$ denote the number of blue points in A_i. We need to prove that $f(i)=2n$ for some i.

   Note first that $|f(i+1)-f(i)|\leq 1$ since A_i and A_{i+1} have $3n-1$ points of S in common. Note also that $f(1)+f(3n+1)=4n$ because the disjoint segments A_1 and A_{3n+1} cover S. If $f(1)=2n$ we are done. So either $f(1)<2n$ and $f(3n+1)>2n$ or $f(1)>2n$ and $f(3n+1)<2n$. Since f increases or decreases by at most 1 when i increases by 1, there must be some i with $1<i<3n+1$ such that $f(i)=2n$.

9. Let $0<x,y,z<1$. Show that:

$$\frac{1}{x(1-y)}+\frac{1}{y(1-z)}+\frac{1}{z(1-x)}\geq 12$$

   Solution: As $0<x<1$ then the AM GM inequality implies:

$$\frac{1}{2}=\frac{1-x+x}{2}\geq\sqrt{x(1-x)}$$

   So that:

$$x(1-x)\leq\frac{1}{4}$$

   Similarly for y and z, so that:

$$\frac{1}{x(1-x)y(1-y)z(1-z)}\geq 4^3$$

   Applying AM GM again:

$$\frac{1}{3}\left(\frac{1}{x(1-y)}+\frac{1}{y(1-z)}+\frac{1}{z(1-x)}\right)\geq\sqrt[3]{\frac{1}{x(1-x)y(1-y)z(1-z)}}\geq 4$$

   From this, the result follows immediately on multiplication of each side by 3.

10. For an integer $r \geq 2$, define $s(r)$ to be the smallest prime number that divides r.

Show that for any integer $n \geq 2$:

$$\sum_{r=2}^{n} s(r) \geq 3n - 5$$

Solution:

For integer $n \geq 2$ let:

$$g(n) = \sum_{r=2}^{n} s(r)$$

with the convention that $g(1) = 0$.

For the inductive step, we note that (for $k \geq 1$)

$$\begin{aligned} s(6k-4) &= 2 \\ s(6k-3) &= 3 \\ s(6k-2) &= 2 \\ s(6k-1) &\geq 5 \\ s(6k) &= 2 \\ s(6k+1) &\geq 5 \end{aligned}$$

Then, for $n \geq 1$, the difference $g(n+6) - g(n)$ has one term from each residue class mod 6, and so:

$$g(n+6) - g(n) \geq 2 + 3 + 2 + 5 + 2 + 5 = 19$$

We proceed by induction on n.

First we verify manually that for $n = 1, \ldots, 6$ we have $g(n) \geq 3n - 5$:

n	$s(n)$	$g(n)$	$3n-5$
1	-	0	-2
2	2	2	1
3	3	5	4
4	2	7	7
5	5	12	10
6	2	14	11

Now suppose that for some n:

$$g(n) \geq 3n - 5$$

Then it follows that:

$$g(n+6) \geq g(n) + 19 \geq 3n + 14 > 3(n+6) - 5$$

also. This proves the statement.