# Mathematical Enrichment Programme
# UCD School of Mathematical Sciences

Dr. Masha Vlasenko

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## Parity

- Can one substitute *s in the expression

$$1 * 2 * 3 * 4 * 5 * 6 * 7 * 8 * 9 * 10 \;=\; 0$$

  with signs + and – so that it would become a valid equality?

- Prove that the number of divisors of a natural number $n$ (including 1 and $n$ itself) is odd if and only if this number is a square.

- Prove that a polynomial with integer coefficients

$$a_0 x^n + a_1 x^{n-1} + \ldots a_{n-1} x + a_n$$

  which assumes odd values at $x = 0$ and $x = 1$ has no integer roots.

## The greatest common divisor and the Euclidean algorithm

- Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every $n = 1, 2, 3, \ldots$

- Let $m, n$ be two integers. Prove that there exist integers $a, b$ such that

$$g.c.d.(m,n) \;=\; am + bn\,.$$

  **Hint:** Use the Euclidean algorithm.

- Let $m, n$ be two integers. Prove that the set of integers which can be written as $am + bn$ with some integer $a, b$ is precisely the set of integer multiples of $g.c.d.(a, b)$.

- Find all pairs of integers $(a, b)$ which solve the following equations:

$$a)\quad 11a + 21b \;=\; 1 \qquad b)\quad 15a + 21b \;=\; 2 \qquad c)\quad 15a + 21b \;=\; 6$$

- Prove the *Chinese Remainder Theorem*: let $m, n$ be coprime and choose any $0 \le a < m$ and $0 \le b < n$; then there exists an integer $k$ whose remainders on divison by $m$ and $n$ are $a$ and $b$ respectively.

- Prove the *Fundamental Theorem of Arithmetic*: every natural number can be uniquely written as a product of primes.

# Divisibility and arithmetic of remainders

- Let $n = \overline{a_k a_{k-1} \ldots a_0}$ be the decimal representation of $n$, that is $n = a_0 + 10a_1 + \ldots + 10^k a_k$. Let $S_n = a_0 + a_1 + \ldots + a_k$, $S'_n = a_0 - a_1 + \ldots + (-1)^k a_k$. Prove that $n$ and $S_n$ have equal remainders on division by 3 and 9; $n$ and $S'_n$ have equal remainders on division by 11.
- Prove that the number $\dfrac{n^5}{5} + \dfrac{n^3}{3} + \dfrac{7n}{15}$ is an integer for every $n = 1, 2, 3, \ldots$
- Let $p$ be a prime and $a$ be a number not divisible by $p$. Prove that the remainders on division by $p$ of the numbers $a, 2a, 3a, \ldots, (p-1)a$ are all different, i.e. every remainder from 1 to $p-1$ occurs exactly once.
- Prove *Fermat's Little Theorem*: for a prime $p$ and any integer $a$, number $a^p$ has the same remainder on division by $p$ as $a$.
- Find the remainder on division by 67 of the number $5^{2014}$.

# Miscellanea

- Prove that if $2^n - 1$ is a prime number, then $n$ is also a prime number.
- With how many zeroes does the product of all numbers from 1 to 100 end?
- Prove that $2^n$ doesn't divide $n!$
- Find the highest power of 2 that divides $(n+1) \cdot (n+2) \cdot \ldots \cdot 2n$.
- Prove that the number $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ is never an integer.