EGMO and UCD Enrichment Programme in Mathematics

Selection Test 16 February 2019

1. The edges of a cube are coloured with three colours such that each vertex is the endpoint of an edge of each of the three colours. Show that there are four parallel edges of the same colour.

2. Let $ABCD$ be a convex quadrilateral. Suppose that $AB = CD$. Prove that
$$BC \cdot (\sin \angle B - \sin \angle C) = AD \cdot (\sin \angle A - \sin \angle D).$$

3. Let a, b, c be the sides of a triangle. Prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

4. Let $ABCD$ be a square. Let P be any point on the circumcircle of $ABCD$ lying on the arc joining A to B (and distinct from A and B). Let M be the point of intersection of DP with the diagonal AC. Let N be the point of intersection of CP with the side AB. Show that MN is parallel to BD.

5. For any positive integer $n \geq 1$ denote $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. Prove that:

   (a) for any integer $n \geq 1$ we have
$$\frac{(2n)^2}{(2n-1)!(2n+1)!} < \frac{1}{(2n-1)!} - \frac{1}{(2n+1)!}.$$

   (b) We have
$$\frac{2^2}{1!3!} + \frac{4^2}{3!5!} + \frac{6^2}{5!7!} + \dots + \frac{2018^2}{2017!2019!} < 1 - \frac{1}{2019!}.$$

6. Finn has 5 distinct real numbers. He takes the sum of each pair of numbers and writes down the 10 sums. The 3 smallest sums are $30, 34$ and 35, while the 2 largest are 46 and 49.

   Determine, with proof, the largest of Finn's 5 numbers.

7. Let ABC be a right-angled triangle with hypotenuse AB. Let D lie on the segment BC with $BD = 2 \cdot DC$. Let M be the midpoint of the hypotenuse AB. Determine the ratio AD/DM.

8. Let S be a set of $6n$ points on a line. $4n$ of these points are painted blue and the other $2n$ points are painted green.

   Prove that there exists a line segment that contains exactly $3n$ points from S, such that $2n$ of them are blue and the other n are green.

9. Let $0 < x, y, z < 1$. Show that:
$$\frac{1}{x(1-y)} + \frac{1}{y(1-z)} + \frac{1}{z(1-x)} \geq 12$$

10. For an integer $r \geq 2$, define $s(r)$ to be the smallest prime number that divides r.

    Show that for any integer $n \geq 2$:
$$\sum_{r=2}^{n} s(r) \geq 3n - 5$$