EGMO and UCD Enrichment Programme in Mathematics
Selection Test 8 February 2020

1. A, B and C are three points on a circle $\mathcal{C}$. Let BD be the bisector of angle $\angle ABC$ and $DE||AB$, where D, E lie also on $\mathcal{C}$. Prove that $|BC| = |DE|$.

2. Let a, b, c be integers such that $a - c$ is even and $b - c$ is divisible by 3. Show that
$$\frac{an^2}{2} + \frac{bn^3}{3} + \frac{cn}{6}$$
is an integer for every integer n.

3. Let $ABCD$ be a convex quadrilateral. Let M, N, O, P be the midpoints of AB, BC, CD and DA respectively.
   (a) Show that $MNOP$ is a parallelogram.
   (b) Show that the area $[MNOP]$ is $1/2$ of the area $[ABCD]$.

4. Alice and Bob play the following game. 2020 coins are placed on a table. The players take turns, each removing either one or two coins in a turn. The player to remove the last coin loses.
   Bob goes first. Which of the two players has a strategy that is guaranteed to win?

5. Let ABC be a triangle. Let $\mathcal{C}_1$ be the circle which passes through A and B and which is tangent to BC. Let $\mathcal{C}_2$ be the circle which passes through A and C and which is tangent to BC. Let T be the point of intersection (other than A) of $\mathcal{C}_1$ and $\mathcal{C}_2$. Show that if $\angle BAT = \angle CAT$ then $|BT| = |CT|$.

6. Note that the integers $6, 10, 15$ have the property that any two of them have a common divisor greater than 1, but the only common divisor of all three is 1.
   (a) Find four integers with the property that any pair of them has a common divisor greater than 1, but no triple of them has a common divisor greater than 1.
   (b) Do there exist 2020 integers with the property that every collection of 1010 of them has a common divisor greater than 1, but no collection of 1011 of them has a common divisor greater than 1?

7. The triple $(1, 5, 7)$ is such that the squares $(1, 25, 49)$ are in arithmetic progression. Show that there are infinitely many triples of positive integers (a, b, c) with greatest common divisor 1 such that a^2, b^2 and c^2 are in arithmetic progression.

8. Let $a, b, c \geq 0$ be real numbers with $a + b + c = 1$.
   Show that:
$$1 \leq \sqrt{a(1+b)} + \sqrt{b(1+c)} + \sqrt{c(1+a)} \leq 2$$

9. Show that there are no integers x, y satisfying $x^2 + xy - 3y^2 = 2020$.

10. A pond has 2020 lily pads arranged in a circle. At time zero, two frogs (Anthony and Clare) share the same lily pad.
    Every minute, Anthony jumps over 99 lily pads in an anti-clockwise direction, to land on a pad 100 removed from where the jump started. At the same time, Clare jumps over 100 lily pads in a clockwise direction, to land on a pad 101 removed from where the jump started.
    What is the first time that Anthony and Clare are again within five lily pads of each other?