COUNTING AND NUMBERING

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International Mathematical Olympiad (IMO) is an annual six-problem contest for pre-collegiate students and is the oldest of the International Science Olympiads.

The first IMO was held in Romania in 1959. It was initially founded for eastern European countries but eventually other countries participated as well. It has since been held annually, except in 1980. About 90 countries send teams of up to six students, plus one team leader, one deputy leader, and observers.

The paper consists of six problems, with each problem being worth seven points, the total score thus being 42 points. No calculators are allowed. The examination is held over two consecutive days; the contestants have four-and-a-half hours to solve three problems per day. The problems chosen are from various areas of secondary school mathematics, broadly classifiable as geometry, number theory, algebra, and combinatorics. They require no knowledge of higher mathematics such as calculus and analysis, and solutions are often short and elementary. However, they are usually disguised so as to make the process of finding the solutions difficult.

Problem 1. (a) How many numbers are in the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191\ ?$$

(b) How many numbers are in the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163\ ?$$

Solution. To answer the above question in a more general framework we need the following definition:

Definition. An **arithmetic progression** or **arithmetic sequence** is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. This difference between any successive terms is called the **ratio** of the arithmetic progression. For instance, the sequence

$$15,\ 16,\ 17,\ \ldots,\ 190,\ 191$$

is an arithmetic progression with ratio 1.

To find the number of the terms in an arithmetic progression we use the formula

$$\frac{\text{last term} - \text{first term}}{\text{ratio}} + 1$$

In our case the total number of terms is

$$\frac{191 - 15}{1} = 176 + 1 = 177 \quad \text{terms}$$

For the second example, the sequence

$$22,\ 25,\ 28,\ 31,\ \ldots,\ 160,\ 163$$

is an arithmetic progression with ratio 2 so the number of terms would be

$$\frac{163-22}{3} = 47 + 1 = 48$$

Let

$$a_1,\ a_2,\ a_3,\ \ldots,\ a_n$$

be an arithmetic progression with n terms and having the ration r. From the above formula we find

$$\frac{a_n - a_1}{r} + 1 = n$$

Hence

$$a_n = a_1 + r(n-1)$$

Another important formula concerns the sum of terms in an arithmetic progression

$$a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}$$

In particular we have

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$

(b) $1 + 3 + 5 + \cdots + (2n-1) = n^2$

Other useful formulas are as follows

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(d) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Problem 2. For any positive integer n find the sum

$$S_n = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$$

Solution. Remark that

$$\begin{aligned}
S_n &= 1(1+1) + 2(2+1) + 3(3+1) + \cdots + n(n+1) \\
&= (1^2+1) + (2^2+2) + (3^2+3) + \cdots + (n^2+n) \\
&= (1^2+2^2+3^3+\cdots+n^2) + (1+2+3+\cdots+n) \\
&= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{2n+1}{3}+1\right] \\
&= \frac{n(n+1)}{2}\frac{2n+4}{3} \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

In the similar way one can compute

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1)(2n+1)$$

Problem 3. For any positive integer n find the sum

$$S_n = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2)$$

Solution. The general term in the above sum is

$$k(k+1)(k+2)$$

where $k = 1, 2, 3, \ldots, n$

Remark that

$$k(k+1)(k+2) = k(k^2 + 3k + 2) = k^3 + 3k^2 + 2k$$

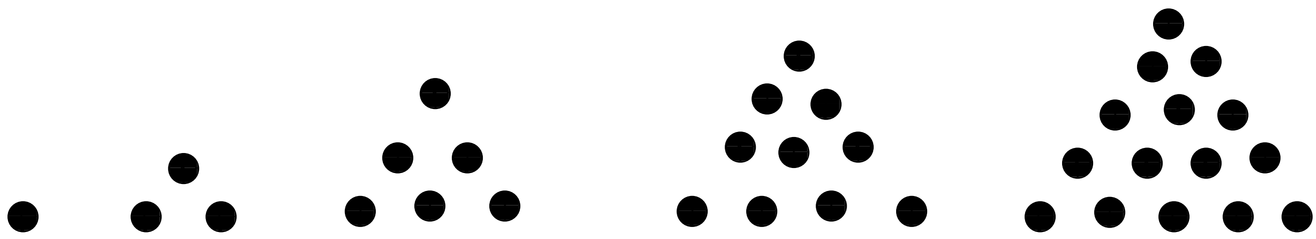
so

$$\begin{aligned}
S_n &= (1^3 + 3 \cdot 1^2 + 2 \cdot 1) + (2^3 + 3 \cdot 2^2 + 2 \cdot 2) + \cdots + (n^3 + 3 \cdot n^2 + 2 \cdot n) \\
&= (1^3 + 2^3 + \cdots + n^3) + 3(1^2 + 2^2 + \cdots + n^2) + 2(1 + 2 + \ldots n) \\
&= \frac{n^2(n+1)^2}{4} + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + (2n+1) + 2\right] \\
&= \frac{n(n+1)}{2}\frac{n^2 + 5n + 6}{2} \\
&= \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}$$

Problem 4. Each of the numbers

$$1 = 1, \quad 3 = 1 + 2, \quad 6 = 1 + 2 + 3, \quad 10 = 1 + 2 + 3 + 4$$

represent the number of balls that can be arranged evenly in an equilateral triangle.



This led the ancient Greeks to call a number **triangular** if it is the sum of consecutive integers beginning with 1.

Prove the following facts about triangular numbers:

(a) If n is a triangular number then $8n + 1$ is a perfect square (Plutarch, circa 100 AD)

(b) The sum of any two triangular numbers is a perfect square (Nicomachus, circa 100 AD)

(b) If n is a triangular number so are the numbers $9n + 1$ and $25n + 3$ (Euler, 1775)

Solution. Remark first that n is a triangular number if there exists a positive integer k such that

$$n = 1 + 2 + 3 + \cdots + k$$

that is,

$$n = \frac{k(k+1)}{2}$$

(a) If $n = \frac{k(k+1)}{2}$ then

$$8n + 1 = 4k(k+1) + 1 = 4k^2 + 4k + 1 = (2k+1)^2$$

(b) Let n and m be two consecutive triangular numbers. Then, there exists $k \geq 1$ such that

$$n = \frac{k(k+1)}{2} \quad \text{and} \quad m = \frac{(k+1)(k+2)}{2}$$

Then

$$n + m = \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2} = \frac{k(k+1) + (k+1)(k+2)}{2}$$

$$n + m = \frac{(k+1)(2k+2)}{2} = (k+1)^2$$

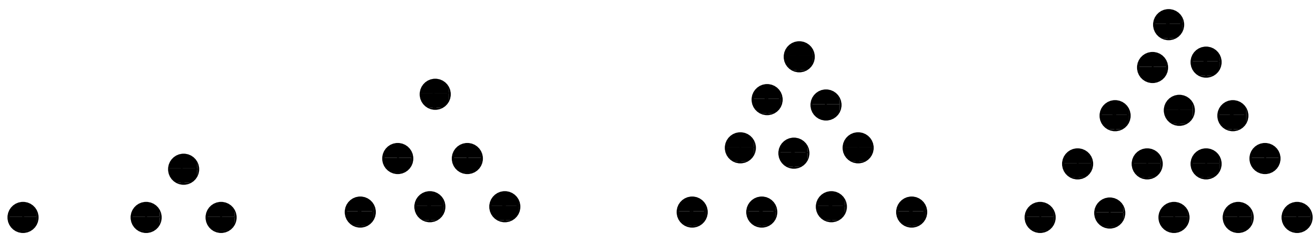
Problem 5. Let t_n be the nth triangular number, that is

$$t_1 = 1, \quad t_2 = 3, \quad t_3 = 6, \quad t_4 = 10, \ldots$$

Prove the formula

$$t_1 + t_2 + \cdots + t_n = \frac{n(n+1)(n+2)}{6}$$

Solution.



We have

$$t_n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}.$$

Therefore,

$$
\begin{aligned}
t_1 + t_2 + \cdots + t_n &= \frac{1^2+1}{2} + \frac{2^2+2}{2} + \frac{3^2+3}{2} + \cdots + \frac{n^2+n}{2} \\
&= \frac{1^2+2^2+3^2+\cdots+n^2}{2} + \frac{1+2+3+\cdots+n}{2} \\
&= \frac{1}{2}\Big[(1^2+2^2+3^2+\cdots+n^2) + (1+2+\cdots+n)\Big] \\
&= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] \\
&= \frac{1}{2}\,\frac{n(n+1)}{2}\left[\frac{2n+1}{3} + 1\right] \\
&= \frac{1}{2}\,\frac{n(n+1)}{2}\frac{2n+4}{3} \\
&= \frac{n(n+1)(2n+4)}{12} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}
$$