Access to Engineering - Mathematics 2
ADEDEX428
Semester 2 2014-2015 Exam Solutions

1. (i) (a)

$$2\begin{pmatrix} -2 & 3 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ 0 & -2 \end{pmatrix}.$$

$$\begin{pmatrix} -3 & 1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \end{pmatrix} \text{ can't be performed.}$$

$$3\begin{pmatrix} 0 & -2 \\ -1 & -3 \\ 2 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 3 \\ -3 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ -3 & -9 \\ 6 & 9 \end{pmatrix} + \begin{pmatrix} -2 & -6 \\ 6 & 2 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & -12 \\ 3 & -7 \\ 6 & 9 \end{pmatrix}.$$

(b) We have $\|(1,-2,2)\| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$.
Hence a unit vector in the direction of $(1,-2,2)$ is $\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$.

(ii) (a)

$$\begin{pmatrix} -2 & -1 \\ -2 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 8 & -17 \\ -3 & 12 \end{pmatrix}.$$

$$\begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 0 & 1 & -5 \end{pmatrix} \text{ can't be performed.}$$

(b)

$$
\begin{aligned}
(3,2,1)\times(-1,-2,-3) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ -1 & -2 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ -2 & -3 \end{vmatrix}\hat{i} - \begin{vmatrix} 3 & 1 \\ -1 & -3 \end{vmatrix}\hat{j} + \begin{vmatrix} 3 & 2 \\ -1 & -2 \end{vmatrix}\hat{k} \\
&=(2\times(-3)-1\times(-2))\hat{i}-(3\times(-3)-1\times(-1))\hat{j} \\
&\quad +(3\times(-2)-2\times(-1))\hat{k} \\
&=-4\hat{i}+8\hat{j}-4\hat{k}.
\end{aligned}
$$

(iii) (a) We will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$.

$$
\begin{array}{rcl}
R2 & \to & R2+R1 \\
R3 & \to & R3-R1
\end{array}
\quad
\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)
$$

$$
R2 \to -R2 \quad
\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)
$$

$$
R3 \to R3+R2 \quad
\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right)
$$

$$
R3 \to -R3 \quad
\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)
$$

$$
R1 \to R1-R3 \quad
\left(\begin{array}{rrr|rrr} 1 & -2 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)
$$

$$
R1 \to R1+2R2 \quad
\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & -3 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)
$$

Hence $\begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -3 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$.

(b) We have

$$\det\left[\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$
$$= (1-\lambda)(2-\lambda) - 2(3)$$
$$= \lambda^2 - 3\lambda + 2 - 6$$
$$= \lambda^2 - 3\lambda - 4$$

Hence the characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda - 4)(\lambda + 1) = 0$. Thus the eigenvalues are $\lambda = 4$ and $\lambda = -1$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = 4$:
We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$.
Thus $\begin{pmatrix} x + 2y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$.
Hence we have the equations $x + 2y = 4x$ and $3x + 2y = 4y$. Both these equations reduce to $3x = 2y$, so taking $y = 3$, say, we obtain the eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$\lambda = -1$:
We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$.
Thus $\begin{pmatrix} x + 2y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.
Hence we have the equations $x + 2y = -x$ and $3x + 2y = -y$. Both these equations reduce to $x = -y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

2. (i)

$$|z| = |1 + 2i| = \sqrt{1^2 + 2^2} = \sqrt{1 + 4} = \sqrt{5}, \quad \overline{z} = \overline{1 + 2i} = 1 - 2i,$$
$$\mathrm{Re}(z) = \mathrm{Re}(1 + 2i) = 1, \quad \mathrm{Im}(z) = \mathrm{Im}(1 + 2i) = 2.$$

$$z + w = (1 + 2i) + (3 - i) = (1 + 3) + (2 - 1)i = 4 + i$$
$$z - w = (1 + 2i) - (3 - i) = (1 - 3) + (2 - (-1))i = -2 + 3i$$
$$zw = (1 + 2i)(3 - i) = ((1)(3) - (2)(-1)) + ((1)(-1) + (2)(3))i = 5 + 5i$$
$$\frac{z}{w} = \frac{1 + 2i}{3 - i} = \frac{1 + 2i}{3 - i} \cdot \frac{3 + i}{3 + i} = \frac{1 + 7i}{10} = \frac{1}{10} + \frac{7}{10}i$$

(ii) The real part of $\sqrt{3} - i$ is positive and its imaginary part is negative, so we are in the situation of Figure 5 in the Complex Numbers notes.

Hence the argument of $\sqrt{3}-i$ is

$$\theta = -\phi = -\tan^{-1}\left(\left|\frac{-1}{\sqrt{3}}\right|\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$

Also, the magnitude of $\sqrt{3}-i$ is $r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$.
Hence, $\sqrt{3}-i$ in polar form is

$$\sqrt{3}-i = 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

To calculate $(\sqrt{3}-i)^4$ we will use Corollary 2.3.9 from the Complex Numbers notes. That is we will use

$$(r(\cos(n\theta) + i\sin(n\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)),$$

with $r = 2$, $\theta = -\dfrac{\pi}{6}$ and $n = 4$.
Hence

$$\begin{aligned}
\left(\sqrt{3}-i\right)^6 &= \left(2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right)^4 \\
&= 2^4\left(\cos\left(-\frac{4\pi}{6}\right) + i\sin\left(-\frac{4\pi}{6}\right)\right) \\
&= 16\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right) \\
&= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) \\
&\simeq -8 - 13.856i.
\end{aligned}$$

(iii) We will use the fact (see P.12 of the Complex Numbers notes) that the nth roots are given by

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \quad k = 0, 1, \ldots, n-1.$$

In this case we have $r = 2$ and $\theta = \dfrac{\pi}{3}$ and we are looking for the fifth roots, so we take $n = 5$.
Thus the roots are

$$z_k = 2^{\frac{1}{5}}\left(\cos\left(\frac{\pi/3}{5} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi/3}{5} + \frac{2k\pi}{5}\right)\right) \quad k = 0, 1, 2, 3, 4.$$

That is

$$z_0 = 2^{\frac{1}{5}}\left(\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right)$$
$$z_1 = 2^{\frac{1}{5}}\left(\cos\left(\frac{7\pi}{15}\right) + i\sin\left(\frac{7\pi}{15}\right)\right)$$
$$z_2 = 2^{\frac{1}{5}}\left(\cos\left(\frac{13\pi}{15}\right) + i\sin\left(\frac{13\pi}{15}\right)\right)$$
$$z_3 = 2^{\frac{1}{5}}\left(\cos\left(\frac{19\pi}{15}\right) + i\sin\left(\frac{19\pi}{15}\right)\right)$$
$$z_4 = 2^{\frac{1}{5}}\left(\cos\left(\frac{25\pi}{15}\right) + i\sin\left(\frac{25\pi}{15}\right)\right)$$

3. (i) For $f(x) = 2x^{-4} - 3x^{\frac{3}{2}}$ we have to use the sum and multiple rule, as well as the derivative of x^n with $n = -4$ and with $n = \frac{3}{2}$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^{-4}\right) + \frac{d}{dx}\left(-3x^{\frac{3}{2}}\right) \\ &= 2\frac{d}{dx}\left(x^{-4}\right) - 3\frac{d}{dx}\left(x^{\frac{3}{2}}\right) \\ &= 2\left(-4x^{-5}\right) - 3\left(\frac{3}{2}x^{\frac{1}{2}}\right) \\ &= -8x^{-5} - \frac{9}{2}x^{\frac{1}{2}}. \end{aligned}$$

For $g(x) = 2\cos(3x) - \sin(-2x)$ we use the derivatives of $\cos(ax)$ and $\sin(ax)$ with $a = 3$ and $a = -2$ together with the sum and multiple rules.

$$g'(x) = 2(-3\sin(3x)) - (-2\cos(-2x)) = -6\sin(3x) + 2\cos(-2x).$$

For $h(x) = \ln\left(\frac{7}{2}x\right) - 4e^{-\frac{5}{2}x}$ we use the derivatives of $\ln(ax)$ with $a = \frac{7}{2}$ and e^{ax} with $a = -\frac{5}{2}$ together with the sum and multiple rules.

$$h'(x) = \frac{1}{x} - 4\cdot\left(-\frac{5}{2}e^{-\frac{5}{2}x}\right) = \frac{1}{x} + 10e^{-\frac{5}{2}x}.$$

(ii) (a) We first have to find the critical points of f. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = -6x^2 + 6x + 36$, so we solve the equation $-6x^2 + 6x + 36 = 0$.
Now

$$\begin{aligned} -6x^2 + 6x + 36 = 0 &\Leftrightarrow x^2 - x - 6 = 0 \\ &\Leftrightarrow (x+2)(x-3) = 0 \\ &\Leftrightarrow x = -2 \text{ or } x = 3. \end{aligned}$$

Thus the critical points are $x = -2$ and $x = 3$.
Next, $f''(x) = -12x + 6$ and we evaluate $f''(x)$ at each of the critical points.
$f''(-2) = -12(-2) + 6 = 30 > 0$, so the critical point at $x = -2$ is a local minimum.
$f''(3) = -12(3)+6 = -30 < 0$, so the critical point at $x = 3$ is a local maximum.

(b) We first have to find the critical points of f. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = 3x^2 - 18x + 24$, so we solve the equation $3x^2 - 18x + 24 = 0$.
Now

$$\begin{aligned} 3x^2 - 18x + 24 = 0 &\Leftrightarrow x^2 - 6x + 8 = 0 \\ &\Leftrightarrow (x-4)(x-2) = 0 \\ &\Leftrightarrow x = 2 \text{ or } x = 4. \end{aligned}$$

Thus the critical points are $x = 2$ and $x = 4$.

We can now find where the global maxima and minima of f occur by evaluating it at the endpoints of the domain and at critical points (since these both lie in the domain).
So we evaluate $f(x)$ at $x = 0, 2, 4, 5$.
$f(0) = -15$, $f(2) = 5$, $f(4) = 1$ and $f(5) = 5$.
Hence the global maximum of f is 5 attained at $x = 2$ and $x = 5$ and the global minimum of f is -15 attained at $x = 0$.

(iii) With $f(x) = \dfrac{e^{2x}\cos(2x)}{x^3\ln(x)}$ we have a product in the numerator and the denominator, so we have to use the product rule twice before we use the quotient rule.
First let us differentiate $g(x) = e^{2x}\cos(3x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{2x}\right)\cos(3x) + e^{2x}\frac{d}{dx}\left(\cos(3x)\right) \\ &= 2e^{2x}\cos(3x) + e^{2x}\left(-3\sin(3x)\right) \\ &= 2e^{2x}\cos(3x) - 3e^{2x}\sin(3x) \\ &= e^{2x}(2\cos(3x) - 3\sin(3x)). \end{aligned}$$

Next we differentiate $h(x) = x^4\ln(x)$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(x^4\right)\ln(x) + x^4\frac{d}{dx}\left(\ln(x)\right) \\ &= 4x^3\ln(x) + x^4\left(\frac{1}{x}\right) \\ &= 4x^3\ln(x) + x^3 \\ &= x^3(4\ln(x) + 1). \end{aligned}$$

We can now use the quotient rule.

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{e^{2x}(2\cos(3x) - 3\sin(3x))x^4\ln(x) - e^{2x}\cos(3x)x^3(4\ln(x)+1)}{\left(x^4\ln(x)\right)^2}. \end{aligned}$$

With $g(x) = \cos(x^3 + 2x^2 - x)$ we will use the chain rule with
$u = x^3 + 2x^2 - x$ and $y = \cos(u)$ (where we are letting $y = g(x)$).
Then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\sin(u)\left(3x^2 + 4x - 1\right) = -\sin\left(x^3 + 2x^2 - x\right)\left(3x^2 + 4x - 1\right).$$

4. (i) (a) For $\int 2x^{-5} - 3x^{\frac{3}{2}}\,dx$, we have to use the sum and multiple rules as well as the integral of x^n with $n = -5$ and $n = \frac{3}{2}$.

$$\begin{aligned}\int 2x^{-5} - 3x^{\frac{3}{2}}\,dx &= \int 2x^{-5}\,dx + \int -3x^{\frac{3}{2}}\,dx \\ &= 2\int x^{-5}\,dx - 3\int x^{\frac{3}{2}}\,dx \\ &= 2\cdot\frac{1}{-4}x^{-4} - 3\cdot\frac{1}{5/2}x^{\frac{5}{2}} + c \\ &= -\frac{1}{2}x^{-4} - \frac{6}{5}x^{\frac{5}{2}} + c.\end{aligned}$$

For $\int -3\sin(-2x) + \cos(4x)\,dx$, we have to use the sum and multiple rules, but this time with the integral of $\sin(ax)$ and the integral of $\cos(ax)$, where we take $a = -2$ and $a = 4$ respectively.

$$\begin{aligned}\int -3\sin(-2x) + \cos(4x)\,dx &= \int -3\sin(-2x)\,dx + \int \cos(4x)\,dx \\ &= -3\int \sin(-2x)\,dx + \int \cos(4x)\,dx \\ &= -3\left(-\frac{1}{-2}\cos(-2x)\right) + \frac{1}{4}\cdot\sin(4x) + c \\ &= -\frac{3}{2}\cos(-2x) + \frac{1}{4}\sin(4x) + c.\end{aligned}$$

(b)

$$\begin{aligned}\int_1^2 4e^{3x} - \frac{2}{x}\,dx &= \left[\frac{4}{3}e^{3x} - 2\ln(x)\right]_1^2 \\ &= \frac{4}{3}e^{6} - 2\ln(2) - \left(\frac{4}{3}e^{3} - 2\ln(1)\right) \\ &= \frac{4}{3}\left(e^{6} - e^{3}\right) - 2\ln(2).\end{aligned}$$

(ii) (a) Since $f(0) = 2$ and since the graph of $f(x) = x^3 - 2x^2 - x + 2$ only crosses the x-axis at $x = -1$ between the points $x = -2$ and $x = 0$, it follows that the graph of $f(x) = x^3 - 2x^2 - x + 2$ lies below the x-axis between the points $x = -2$ and $x = -1$ and above the x-axis between the points $x = -1$ and $x = 0$.

Hence the required area is

$$
\begin{aligned}
&-\int_{-2}^{-1} x^3-2x^2-x+2\,dx+\int_{-1}^{0} x^3-2x^2-x+2\,dx\\
&\quad=\left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{1}{2}x^2+2x\right]_{-2}^{-1}+\left[\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{1}{2}x^2+2x\right]_{-1}^{0}\\
&\quad=-\left[\left(\frac{1}{4}(-1)^4-\frac{2}{3}(-1)^3-\frac{1}{2}(-1)^2+2(-1)\right)\right.\\
&\qquad\quad\left.-\left(\frac{1}{4}(-2)^4-\frac{2}{3}(-2)^3-\frac{1}{2}(-2)^2+2(-2)\right)\right]\\
&\qquad+\left[\left(\frac{1}{4}0^4-\frac{2}{3}0^3-\frac{1}{2}0^2+2(0)\right)-\left(\frac{1}{4}(-1)^4-\frac{2}{3}(-1)^3-\frac{1}{2}(-1)^2+2(-1)\right)\right]\\
&\quad=-\left[-\frac{19}{12}-\frac{10}{3}\right]+\left[0-\left(-\frac{19}{12}\right)\right]\\
&\quad=\frac{13}{2}.
\end{aligned}
$$

(b) Using the formula $V=\pi\int_a^b f(x)^2\,dx$, the volume is

$$
\begin{aligned}
V&=\pi\int_0^{\pi}\cos^2(x)\,dx\\
&=\pi\int_0^{\pi}\frac{1}{2}+\frac{1}{2}\cos(2x)\,dx\\
&=\pi\left[\frac{1}{2}x+\frac{1}{4}\sin 2x\right]_0^{\pi}\\
&=\pi\left[\left(\frac{1}{2}\pi+0\right)-(0+0)\right]\\
&=\frac{\pi^2}{2}.
\end{aligned}
$$

(iii) (a) Here we use integration by parts.
Let $f(x)=-2x$ and $g'(x)=\cos(-3x)$, so that $f'(x)=-2$ and $g(x)=\frac{1}{-3}\sin(-3x)=-\frac{1}{3}\sin(-3x)$.
Hence, using the integration by parts formula,

$$
\begin{aligned}
\int -2x\cos(-3x)\,dx&=-2x\left(-\frac{1}{3}\sin(-3x)\right)-\int -2\left(-\frac{1}{3}\sin(-3x)\right)\,dx\\
&=\frac{2}{3}x\sin(-3x)-\int\frac{2}{3}\sin(-3x)\,dx.\\
&=\frac{2}{3}x\sin(-3x)-\frac{2}{3}\left(-\frac{1}{-3}\cos(-3x)\right)+c\\
&=\frac{2}{3}x\sin(-3x)-\frac{2}{9}\cos(-3x)+c.
\end{aligned}
$$

(b) Here we use integration by substitution.
Let $u = 2x^3 + 3$, so that so that $\frac{du}{dx} = 6x^2$.
Then $dx = \frac{du}{du/dx} = \frac{du}{6x^2}$.
Also, when $x = -1$, $u = 1$ and when $x = 1$, $u = 5$.
Hence

$$\begin{aligned}\int_{-1}^{1} x^2(2x^3+3)^6\,dx &= \int_1^5 x^2u^6 \cdot \frac{du}{6x^2}\\ &= \int_1^5 \frac{1}{6}u^6\,du\\ &= \left[\frac{1}{42}u^7\right]_1^5\\ &= \frac{1}{42}(5^7 - 1^7)\\ &= \frac{39062}{21}.\end{aligned}$$

5. (i) (a) Let A be the event 'the staff member is a tutor' and let B be the event 'the staff member is male', so that $P(A \cup B)$ is the probability that the staff member is either a tutor or is male. Since there are 35 tutors out of a total of 55 staff members, $P(A) = \frac{35}{55}$. Also, there are 12 male lecturers and $35 - 17 = 18$ male tutors making a total of 30 male staff members, so $P(B) = \frac{30}{55}$. Next, since there are 18 male tutors, $P(A \cap B) = \frac{18}{55}$. Hence we have that the probability that the staff member selected is either a tutor or is male is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{35}{55} + \frac{30}{55} - \frac{18}{55} = \frac{47}{55}.$$

(b) Let A be the event that a customer likes dark chocolate and let B be the event that a customer likes milk chocolate. Now, the probability we want is $P(A|B)$ and we are given in the question that $P(B) = 0.75$ and that $P(A \cap B) = 0.55$. Hence if we pick a customer who likes milk chocolate at random, the probability that they also like dark chocolate is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.55}{0.75} = \frac{11}{15}.$$

(ii) Let us call tossing a head a success and let X denote the number of successes we get in fifteen tosses, so that we want to find $P(X = 9)$. Since $n = 15$, $k = 9$, $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$, the required probability is

$$P(X = 9) = \binom{15}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{15-9} = \frac{5005}{32768}.$$

(iii) We will assume that this is a Poisson process with $\lambda = 6$, where λ is the average number of cars arriving in the thirty minutes between 5.30 pm and 6 pm on Sunday. Thus we have to calculate $P(X \geqslant 5)$, where X is the number of customers arriving in the given thirty minutes. However $P(X \geqslant 5) = 1 - P(X \leqslant 4)$ and using the Poisson distribution tables we see that $P(X \leqslant 4) \simeq 0.2851$. Thus the required probability is

$$P(X \geqslant 5) = 1 - P(X \leqslant 4) \simeq 1 - 0.2851 = 0.7149.$$