University College Dublin
An Coláiste Ollscoile, Baile Átha Cliath

SPECIMEN EXAMINATION 2013/2014

ADEDEX428

Mathematics for Engineering

Dr. Anthony Cronin

Dr. Anthony Brown*

Time Allowed: 3 hours

Instructions for Candidates

Candidates should attempt all questions.
Note that different questions have different marks.

Note for Invigilators

Non programmable calculators are permitted
Graph paper is not required

1. (i) (a) Determine if the following can be formed and calculate them if they can.

$$2\begin{pmatrix} 2 & 5 \\ -3 & 0 \end{pmatrix}, \quad 3\begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} - 2\begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}$$

$$\text{and} \quad 2\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix}.$$

[6]

(b) Find the angle (in radians to 2 decimal places) between the vectors $(1,3)$ and $(3,1)$. **[2]**

(ii) (a) Determine if the following can be formed and calculate them if they can.

$$\begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 9 \end{pmatrix}^T.$$

[6]

(b) Find the cross product $(1,2,3) \times (-3,-2,-1)$. **[4]**

(iii) (a) Using row reduction, determine if the matrix $\begin{pmatrix} 1 & -2 & 6 \\ -4 & 9 & -23 \\ -1 & 2 & -5 \end{pmatrix}$ has an inverse and find it if it exists. **[6]**

(b) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} 13 & 14 \\ -7 & -8 \end{pmatrix}$.

[6]

2. (i) For $z = 2 - 3i$ and $w = -1 + 5i$, calculate
$|z|, \overline{z}, \mathrm{Re}(z), \mathrm{Im}(z), z+w, z-w, zw$ and $\dfrac{z}{w}$. **[5]**

(ii) Convert $3\sqrt{3} - 3i$ into polar form and hence calculate $(3\sqrt{3} - 3i)^5$, expressing your final answer both in exact polar form and in Cartesian form to three decimal places.

[5]

(iii) Given that $1 + \sqrt{3}i = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$, calculate all the fifth roots of $1 + \sqrt{3}i$, leaving your answers in polar form. **[5]**

3. (i) Differentiate the functions

$$\begin{aligned} f(x) &= 2x^{-6} - 3x^{\frac{7}{3}}, \\ g(x) &= -2\cos(2x) - \sin(-3x) \\ \text{and} \quad h(x) &= \ln\left(\frac{7}{3}x\right) - e^{\frac{7}{3}x}. \end{aligned}$$

[6]

(ii) (a) Starting with the initial guess $x_0 = 2$, apply two iterations of the Newton-Raphson method to obtain an approximate solution of the equation $x^3 - 9x^2 + 12x + 10 = 0$. **[4]**

(b) Find the points where the global maximum and minimum of the function

$$\begin{aligned} f\colon [0,6] &\to \mathbb{R} \\ x &\mapsto x^3 - 9x^2 + 24x - 15 \end{aligned}$$

occur. **[4]**

(iii) Differentiate the functions

$$\begin{aligned} f(x) &= \frac{x^2\cos(3x)}{e^{2x}\sin(x)} \text{ (where } \sin(x) \neq 0) \\ \text{and} \quad g(x) &= \cos(x^3 - 2x + 5). \end{aligned}$$

[6]

4. (i) (a) Find

$$\int 2x^{-6} - 3x^{\frac{7}{3}}\,dx$$

$$\text{and} \int -2\cos(2x) - \sin(-3x)\,dx.$$

[4]

(b) Evaluate $\displaystyle\int_1^2 \frac{2}{x} + 2e^{-3x}\,dx$. **[2]**

(ii) (a) Find the area lying between the graph of $f(x) = x^3 - 2x^2 - 5x + 6$ and the x-axis between the points $x = 0$ and $x = 2$. Hint: The graph of this function only crosses the x-axis at $x = 1$ in the interval $[0, 2]$. **[3]**

(b) Find the volume of revolution of the function $f(x) = \sqrt{1 - x^2}$ about the x-axis between $x = -1$ and $x = 1$. **[3]**

(iii) (a) Find $\displaystyle\int 2xe^{3x}\,dx$. **[4]**

(b) Evaluate $\displaystyle\int_{-1}^{1} x^3(3x^4 - 3)^8\,dx$. **[4]**

5. (i) (a) In a group of 90 undergraduate students at UCD, there are 57 first years, 46 females and 23 female first years. If one of these students is selected at random, what is the probability that they will either be a first year or male? **[3]**

(b) In a supermarket survey, it was found that the probability that a customer likes chocolate ice cream is 0.65 and the probability that a customer likes both chocolate ice cream and strawberry ice cream is 0.4. What is the probability that someone who likes chocolate ice cream also likes strawberry ice cream? **[3]**

(ii) The number of births per week in a particular town is four. What is the probability that there will be exactly three births in the next two days? **[4]**

(iii) A particular brand of light bulb is known to have a life normally distributed with mean 1375 hours and standard deviation 100 hours. What is the probability of a randomly selected bulb of this brand lasting between 1200 hours and 1500 hours?

[5]

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Formula Sheet

Matrices and Vectors

Matrix inverses: If $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Length of a vector: If $\mathbf{a} = (a_1, a_2, a_3)$, then $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Dot product: If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Angle between vectors: $\theta = \cos^{-1}\left(\dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}\right)$.

Cross product: $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$, where $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$.

Eigenvalues: To find the eigenvalues of A, solve $\det(A - \lambda I) = 0$.

Eigenvectors: To find the eigenvector $\mathbf{v}$ corresponding to the eigenvalue λ, solve $A\mathbf{v} = \lambda\mathbf{v}$, where $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Complex numbers

Real part of a complex number: $\mathrm{Re}(a + bi) = a$.

Imaginary part of a complex number: $\mathrm{Im}(a + bi) = b$.

Modulus of a complex number: $|a + bi| = \sqrt{a^2 + b^2}$.

Complex conjugate of a complex number: $\overline{a + bi} = a - bi$.

Dividing complex numbers: $\dfrac{a+bi}{c+di} = \dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di}$.

Polar form: $a + bi = r(\cos(\theta) + i\sin(\theta))$, where $r = |a + bi|$ and we find θ as follows.

First calculate $\phi = \tan^{-1}\left(\left|\dfrac{b}{a}\right|\right)$. Then

$$\theta = \phi \text{ if } a > 0 \text{ and } b > 0, \qquad \theta = \pi - \phi \text{ if } a < 0 \text{ and } b > 0,$$

$$\theta = \phi - \pi \text{ if } a < 0 \text{ and } b < 0 \quad \text{and} \quad \theta = -\phi \text{ if } a > 0 \text{ and } b < 0.$$

De Moivre's formula: $[r(\cos(\theta) + i\sin(\theta))]^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
Roots of complex numbers: The n'th roots of $r(\cos(\theta) + i\sin(\theta))$ are

$$z_k = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \qquad k = 0, 1, \ldots, n-1.$$

Differential Calculus

$f(x)$	$f'(x)$	Comments
c	0	Here c is any real number
x^n	nx^{n-1}	
e^{ax}	ae^{ax}	
$\ln(ax)$	$\frac{1}{x}$	Here we must have $ax > 0$
$\sin(ax)$	$a\cos(ax)$	
$\cos(ax)$	$-a\sin(ax)$	Note the change of sign

Table 1: Some common derivatives

The product rule for differentiation: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.
The quotient rule for differentiation: $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$.
The chain rule for differentiation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.
The critical points of a function f occur at points x where $f'(x) = 0$.
Classifying critical points: Local minima occur where $f''(x) > 0$.
Local maxima occur where $f''(x) < 0$.
The Newton-Raphson method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Integral Calculus

Evaluating definite integrals: If F is an antiderivative of f then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Integration by substitution: $dx = \frac{du}{du/dx}$.
Integration by parts: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$.
Volume of solid of revolution: $V = \pi \int_a^b f(x)^2\,dx$.

$f(x)$	$\int f(x)\,dx$	Comments
k	$kx + c$	Here k is any real number
x^n	$\frac{1}{n+1}x^{n+1} + c$	Here we must have $n \neq -1$
$\frac{1}{x}$	$\ln(x) + c$	Here we must have $x > 0$
e^{ax}	$\frac{1}{a}e^{ax} + c$	
$\sin(ax)$	$-\frac{1}{a}\cos(ax) + c$	Note the change of sign
$\cos(ax)$	$\frac{1}{a}\sin(ax) + c$	

Table 2: Some common integrals

Probability

Sum rule for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Conditional probability: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

Binomial distribution: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Poisson distribution: $P(X = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$.

Normal distribution: $P(a \leqslant X \leqslant b) = P\left(\dfrac{a-\mu}{\sigma} \leqslant Z \leqslant \dfrac{b-\mu}{\sigma}\right)$.