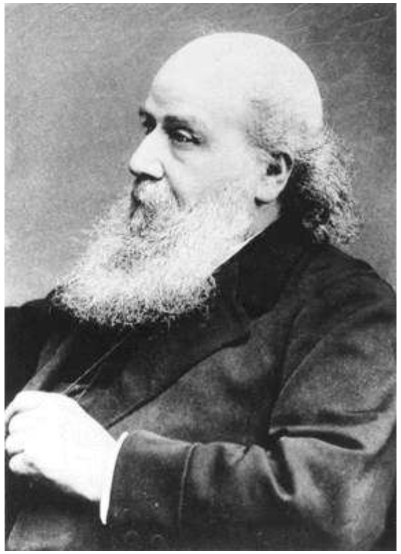
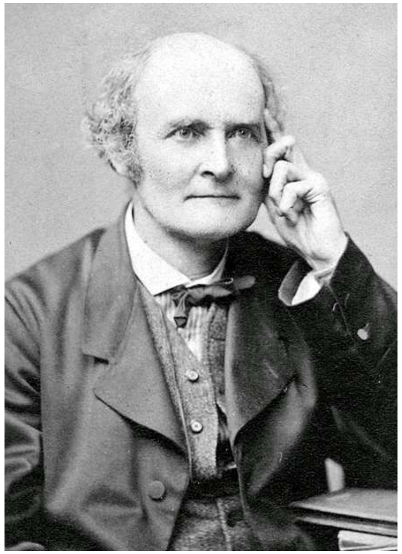
Introduction to Matrices

The happy founders of modern matrix theory: Sylvester, Cayley and Hamilton.



James Joseph Sylvester
1814 – 1897



Arthur Cayley
1821 – 1895

William Rowan Hamilton
1805 – 1865

Matrix addition and subtraction: Example 1.1.6

Find $A + B$ when

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

A and B are 2×4 matrices so $A + B$ is also a 2×4 matrix. Add together the corresponding entries in A and B to get the entries for $A + B$.

$$A + B = \begin{pmatrix} \quad \end{pmatrix}$$

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$$A + B = \begin{pmatrix} 1 & & & \\ & & & \end{pmatrix}$$

Matrix addition and subtraction: Example 1.1.6

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$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

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Matrix addition and subtraction: Example 1.1.6

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$$A + B = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & & & \end{pmatrix}$$

Matrix addition and subtraction: Example 1.1.6

Find $A + B$ when

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$$A + B = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & & \end{pmatrix}$$

Matrix addition and subtraction: Example 1.1.6

Find $A + B$ when

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

A and B are 2×4 matrices so $A + B$ is also a 2×4 matrix. Add together the corresponding entries in A and B to get the entries for $A + B$.

$$A + B = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & 5 & \end{pmatrix}$$

Matrix addition and subtraction: Example 1.1.6

Find $A + B$ when

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

A and B are 2×4 matrices so $A + B$ is also a 2×4 matrix. Add together the corresponding entries in A and B to get the entries for $A + B$.

$$A + B = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & 5 & 3 \end{pmatrix}$$

Matrix addition and subtraction: Example 1.1.6

Find $A + B$ when

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}.$$

A and B are 2×4 matrices so $A + B$ is also a 2×4 matrix. Add together the corresponding entries in A and B to get the entries for $A + B$.

$$A + B = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & 5 & 3 \end{pmatrix}$$

$A - B$ is calculated similarly by subtracting the corresponding entries.

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Matrix multiplication: Example 1.2.2

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Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} & \\ & \end{pmatrix}$$

Consider the top left entry $(AB)_{11}$ ($i = 1$, $j = 1$).

$(AB)_{11} =$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} & \\ & \end{pmatrix}$$

Consider the top left entry $(AB)_{11}$ ($i = 1$, $j = 1$).

$(AB)_{11} = 2 \times 3$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} & \\ & \end{pmatrix}$$

Consider the top left entry $(AB)_{11}$ ($i = 1$, $j = 1$).

$(AB)_{11} = \ 2 \times 3 \ + -1 \times 1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} & \\ & \end{pmatrix}$$

Consider the top left entry $(AB)_{11}$ ($i = 1$, $j = 1$).

$(AB)_{11} = \; 2 \times 3 \; + \; -1 \times 1 \; + \; 3 \times 0$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

Consider the top left entry $(AB)_{11}$ ($i = 1$, $j = 1$).

$(AB)_{11} = \ 2 \times 3 \ + \ -1 \times 1 \ + \ 3 \times 0 \ = 6 + -1 + 0 = 5$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

Consider the top right entry $(AB)_{12}$ ($i = 1$, $j = 2$).

$(AB)_{12} =$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

Consider the top right entry $(AB)_{12}$ ($i = 1$, $j = 2$).

$(AB)_{12} = 2 \times 1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

Consider the top right entry $(AB)_{12}$ ($i = 1$, $j = 2$).

$(AB)_{12} = \; 2 \times 1 \; + \; -1 \times -1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & \\ & \end{pmatrix}$$

Consider the top right entry $(AB)_{12}$ ($i = 1$, $j = 2$).

$(AB)_{12} = \; 2 \times 1 \; + \; -1 \times -1 \; + \; 3 \times 2$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ & \end{pmatrix}$$

Consider the top right entry $(AB)_{12}$ ($i = 1$, $j = 2$).

$$(AB)_{12} = \; 2 \times 1 \; + \; -1 \times -1 \; + \; 3 \times 2 \;\; = 2 + 1 + 6 \; = \; 9$$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ \blacksquare & \end{pmatrix}$$

Consider the bottom left entry $(AB)_{21}$ ($i = 2$, $j = 1$).

$(AB)_{21} =$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ & \end{pmatrix}$$

Consider the bottom left entry $(AB)_{21}$ ($i = 2$, $j = 1$).

$(AB)_{21} = 1 \times 3$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ \quad & \end{pmatrix}$$

Consider the bottom left entry $(AB)_{21}$ ($i = 2$, $j = 1$).

$(AB)_{21} = \ 1 \times 3 \ + \ 0 \times 1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ & \end{pmatrix}$$

Consider the bottom left entry $(AB)_{21}$ ($i = 2$, $j = 1$).

$$(AB)_{21} = 1 \times 3 + 0 \times 1 + -1 \times 0$$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & \end{pmatrix}$$

Consider the bottom left entry $(AB)_{21}$ ($i = 2$, $j = 1$).

$$(AB)_{21} = \ 1 \times 3 \ + \ 0 \times 1 \ + \ -1 \times 0 \ = 3 + 0 + 0 = 3$$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & \end{pmatrix}$$

Finally, consider the bottom right entry $(AB)_{22}$ ($i = 2$, $j = 2$).

$(AB)_{22} =$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & \end{pmatrix}$$

Finally, consider the bottom right entry $(AB)_{22}$ ($i = 2$, $j = 2$).

$(AB)_{22} = 1 \times 1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & \end{pmatrix}$$

Finally, consider the bottom right entry $(AB)_{22}$ ($i = 2$, $j = 2$).

$(AB)_{22} = 1 \times 1 + 0 \times -1$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & \end{pmatrix}$$

Finally, consider the bottom right entry $(AB)_{22}$ ($i = 2$, $j = 2$).

$(AB)_{22} = 1 \times 1 + 0 \times -1 + -1 \times 2$

Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & -1 \end{pmatrix}$$

Finally, consider the bottom right entry $(AB)_{22}$ ($i = 2$, $j = 2$).

$$(AB)_{22} = \; 1 \times 1 \; + \; 0 \times -1 \; + \; -1 \times 2 \; = 1 + 0 + -2 \; = \; -1$$

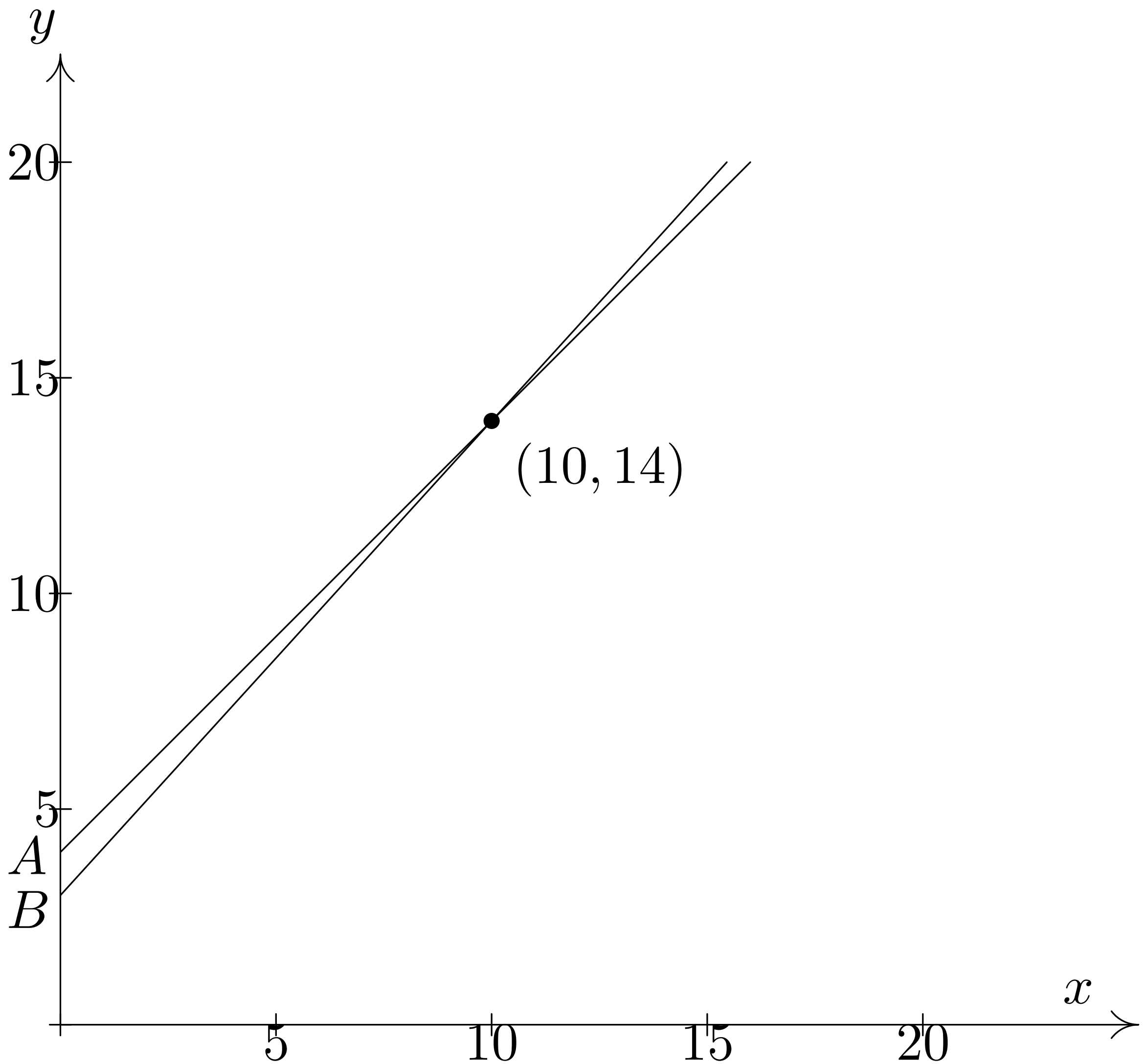
Matrix multiplication: Example 1.2.2

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

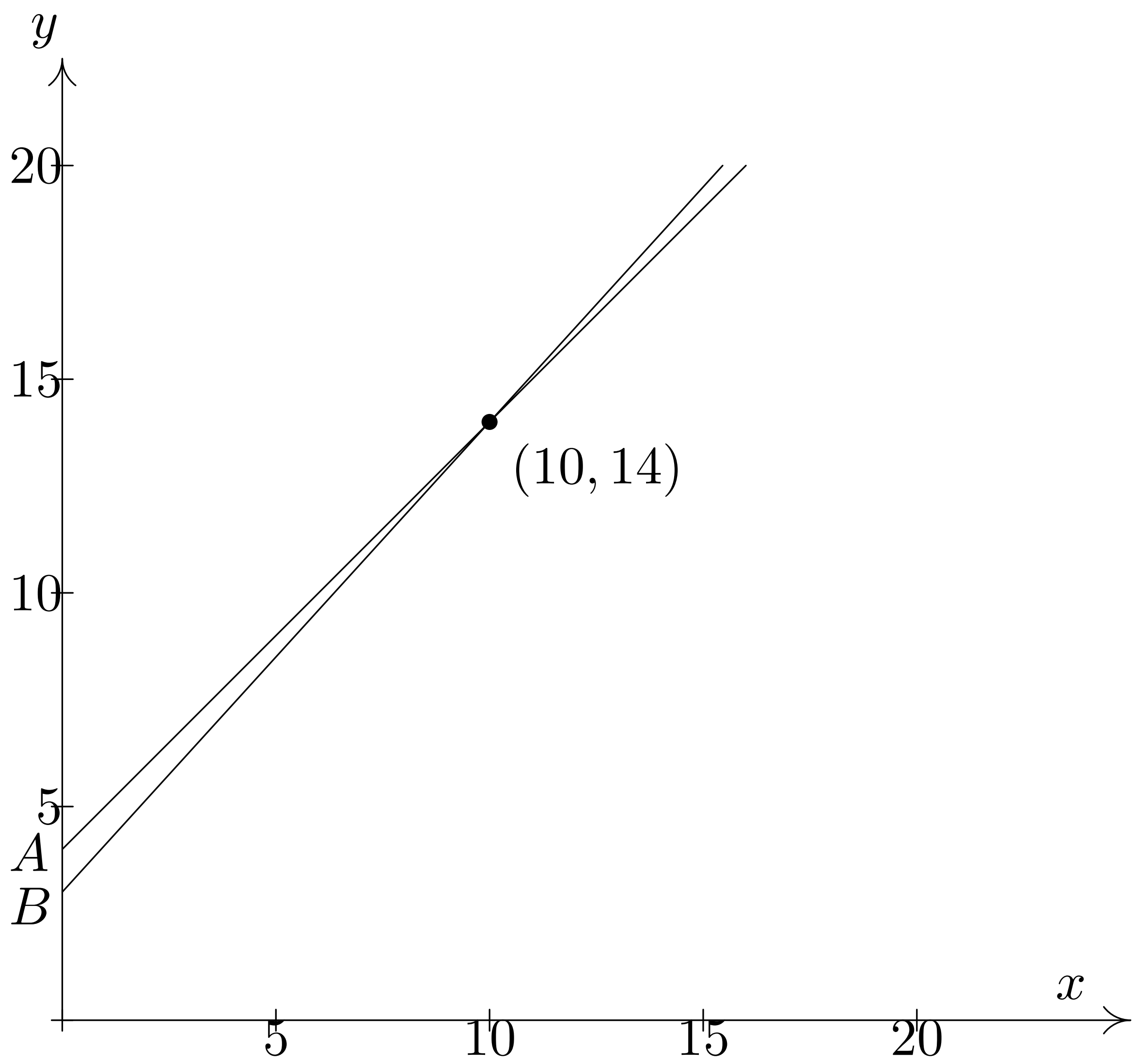
Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & -1 \end{pmatrix}$$

Taxis: Example 1.4.2



Taxis: Example 1.4.2



The price lines appear to cross at $x = 10$ and $y = 14$. It is better to use company B for journeys of less than 10 miles, and company A for journeys of more than 10 miles.

Definition 1.4.5: Elementary Row Operations

Operations on a matrix of the following 3 types are called *Elementary Row Operations* (EROs).

1. Multiply a row by a non-zero constant.
2. Add a multiple of one row to another row.
3. We also allow operations of the following type: interchange two rows in the matrix (this only amounts to writing down the equations of the system in a different order).

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \rightarrow R3 + R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \to R3 + R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$
2. $R2 \to R2 - 3R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & - & 5y & + & z & = & -6 \\ & & 6y & + & z & = & 5 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ -x & + & 4y & + & 2z & = & 0 \end{array}$
1. $R3 \to R3 + R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ 3x & + & y & - & 2z & = & 9 \\ & & 6y & + & z & = & 5 \end{array}$
2. $R2 \to R2 - 3R1$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & -5 & 1 & -6 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & - & 5y & + & z & = & -6 \\ & & 6y & + & z & = & 5 \end{array}$
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$
4. $R3 \to R3 - 6R2$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -11 & 11 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & -11z & = & 11 \end{array}$

Applying EROs to solve Example 1.4.3

ERO	Matrix	System
3. $R2 \to R2 + R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 6 & 1 & 5 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & 6y & + & z & = & 5 \end{array}$
4. $R3 \to R3 - 6R2$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -11 & 11 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & -11z & = & 11 \end{array}$
5. $R3 \to -\frac{1}{11} \times R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.3

ERO	Matrix	System
5. $R3 \to -\frac{1}{11} \times R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{aligned} x + 2y - z &= 5 \\ y + 2z &= -1 \\ z &= -1 \end{aligned}$

Reduced Row-Echelon Form (RREF) of Example 1.4.3

ERO	Matrix	System
5. $R3 \to -\frac{1}{11} \times R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$
6. $R1 \to R1 - 2R2$	$\begin{pmatrix} 1 & 0 & -5 & 7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & & & - & 5z & = & 7 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.3

ERO	Matrix	System
5. $R3 \to -\frac{1}{11} \times R3$	$\left(\begin{array}{cccc} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$
6. $R1 \to R1 - 2R2$	$\left(\begin{array}{cccc} 1 & 0 & -5 & 7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right)$	$\begin{array}{rcrcrcr} x & & & - & 5z & = & 7 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$
7. $R2 \to R2 - 2R3$	$\left(\begin{array}{cccc} 1 & 0 & -5 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$	$\begin{array}{rcrcrcr} x & & & - & 5z & = & 7 \\ & & y & & & = & 1 \\ & & & & z & = & -1 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.3

ERO	Matrix	System
5. $R3 \to -\frac{1}{11} \times R3$	$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 5 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$
6. $R1 \to R1 - 2R2$	$\begin{pmatrix} 1 & 0 & -5 & 7 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & & & - & 5z & = & 7 \\ & & y & + & 2z & = & -1 \\ & & & & z & = & -1 \end{array}$
7. $R2 \to R2 - 2R3$	$\begin{pmatrix} 1 & 0 & -5 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & & & - & 5z & = & 7 \\ & & y & & & = & 1 \\ & & & & z & = & -1 \end{array}$
8. $R1 \to R1 + 5R3$	$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & & & & & = & 2 \\ & & y & & & = & 1 \\ & & & & z & = & -1 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.7

This slide relates to Examples 1.4.7, 1.4.9 and 1.4.10.

Matrix	System
$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$	$\begin{array}{rcrcrcrcl} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.7

This slide relates to Examples 1.4.7, 1.4.9 and 1.4.10.

	Matrix	System
	$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$	$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array}$
REF	$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ & & x_2 & + & 4x_3 & + & 3x_4 & = & 10 \\ & & & & x_3 & + & x_4 & = & 2 \end{array}$

Reduced Row-Echelon Form (RREF) of Example 1.4.7

This slide relates to Examples 1.4.7, 1.4.9 and 1.4.10.

	Matrix	System
	$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$	$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 \end{array}$
REF	$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 \\ & & x_2 & + & 4x_3 & + & 3x_4 & = & 10 \\ & & & & x_3 & + & x_4 & = & 2 \end{array}$
RREF	$\begin{pmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$	$\begin{array}{rrrcrcr} x_1 & & & + & 2x_4 & = & 4 \\ & x_2 & & - & x_4 & = & 2 \\ & & x_3 & + & x_4 & = & 2 \end{array}$

Inconsistent systems: Example 1.4.12

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 7 & 13 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 5 \\ 2x & + & 5y & + & 7z & = & 13 \\ & & 3y & + & 3z & = & 10 \end{array}$

Inconsistent systems: Example 1.4.12

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 7 & 13 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 5 \\ 2x & + & 5y & + & 7z & = & 13 \\ & & 3y & + & 3z & = & 10 \end{array}$
1. $R2 \to R2 - 2R1$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	

Inconsistent systems: Example 1.4.12

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 7 & 13 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 5 \\ 2x & + & 5y & + & 7z & = & 13 \\ & & 3y & + & 3z & = & 10 \end{array}$
1. $R2 \to R2 - 2R1$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	
2. $R3 \to R3 - 3R2$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	

Inconsistent systems: Example 1.4.12

ERO	Matrix	System
	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 5 & 7 & 13 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 5 \\ 2x & + & 5y & + & 7z & = & 13 \\ & & 3y & + & 3z & = & 10 \end{array}$
1. $R2 \to R2 - 2R1$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 3 & 10 \end{pmatrix}$	
2. $R3 \to R3 - 3R2$	$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	

The last line of the third matrix corresponds to the equation

$$0x + 0y + 0z = 1.$$

This has *no* solutions. The original system is *inconsistent*.

Solving Systems of Linear Equations: Summary

1. **Unique Solution**

This will happen if the system is consistent and all variables are leading variables. In this case the RREF obtained from the augmented matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with possibly some additional rows consisting entirely of 0s at the bottom. The unique solution can be read from the rightmost column. See the RREF from Example 1.4.3.

Solving Systems of Linear Equations: Summary

2. **Infinitely Many Solutions**

This happens if the system is consistent but at least one of the variables is free. In this case the system has a *general solution* with at least one parameter. See Examples 1.4.7, 1.4.9 and 1.4.10.

Solving Systems of Linear Equations: Summary

2. **Infinitely Many Solutions**

This happens if the system is consistent but at least one of the variables is free. In this case the system has a *general solution* with at least one parameter. See Examples 1.4.7, 1.4.9 and 1.4.10.

3. **No Solutions**

The system may be inconsistent, i.e., it has no solutions. This happens if a REF obtained from the augmented matrix has a leading 1 in its rightmost column.

$$\begin{pmatrix} & & \vdots & & & \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

See Example 1.4.12.