ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 2

MATH00040

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1. Matrices and Vectors

1.1. **Introduction to Matrices**.

The modern theory of matrices began with the work of Sylvester, Cayley and Hamilton.

Definition 1.1.1. A $m \times n$ ('m by n') *matrix* is a rectangle of numbers having m rows and n columns, enclosed by brackets.

Example 1.1.2.

$$\begin{pmatrix} 2 & 3 & -1 \\ -3 & -\frac{5}{7} & 0 \end{pmatrix} \text{ is a } 2 \times 3 \text{ matrix and } \begin{pmatrix} 2 & 3 \\ 2 & 7 \\ 4 & 0 \end{pmatrix} \text{ is a } 3 \times 2 \text{ matrix.}$$

Remark 1.1.3.

(1) Two matrices are said to have the *same size* if they have the same number of rows and the same number of columns, so a 2×3 matrix and a 3×2 matrix are considered to be of different size.
(2) A matrix having just one row is sometimes called a *row vector*, and a matrix having just one column is called a *column vector*.
(3) We usually label matrices using capital letters like A, B, C, etc.
(4) If a $m \times n$ matrix has the same number of rows as columns, i.e., if $m = n$, then it is called a *square matrix*.
(5) If A is a $m \times n$ matrix, the number appearing in the ith row and jth column is called the (i, j) *entry* of A, and denoted a_{ij}.
(6) Two matrices are *equal* if and only if they have the same size and the corresponding entries are all equal.

Example 1.1.4. If $A = \begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \end{pmatrix}$ then $a_{12} = 3$ $(i = 1,\ j = 2)$, $a_{23} = 5$ $(i = 2,\ j = 3)$, $a_{13} = -1$, etc.

Definition 1.1.5 (Matrix Addition)**.** Let A and B be matrices of the same size ($m \times n$). We define their *sum* $A+B$ to be the $m \times n$ matrix whose entries are given by

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$

Thus $A+B$ is obtained from A and B by adding entries in corresponding positions.

Example 1.1.6. On slides.
Let $A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 1 & 2 & 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 & 0 & -2 \\ 3 & -3 & 1 & 1 \end{pmatrix}$ (A and B are both 2×4 matrices). Then

$$A + B = \begin{pmatrix} 2+(-1) & 0+1 & -1+0 & -1+(-2) \\ 1+3 & 2+(-3) & 4+1 & 2+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & -3 \\ 4 & -1 & 5 & 3 \end{pmatrix}$$

Subtraction of matrices is now defined in the obvious way - e.g., with A and B as in Example 1.1.6, we have

$$A - B = \begin{pmatrix} 2-(-1) & 0-1 & -1-0 & -1-(-2) \\ 1-3 & 2-(-3) & 4-1 & 2-1 \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 & 1 \\ -2 & 5 & 3 & 1 \end{pmatrix}$$

We can only add or subtract two matrices if they are the same size. If A and B are different sizes then the matrix sum $A+B$ is *undefined*.

Definition 1.1.7 (Multiplication of a Matrix by a Real Number)**.** Let A be a $m \times n$ matrix and let k be a real number. Then kA is the $m \times n$ matrix with entries defined by

$$(kA)_{ij} = ka_{ij}$$

i.e. kA is obtained from A by multiplying every entry in A by k.

Example 1.1.8. If $A = \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}$, then

$$2A = \begin{pmatrix} 4 & 2 \\ 6 & -8 \end{pmatrix}, \ -3A = \begin{pmatrix} -6 & -3 \\ -9 & 12 \end{pmatrix}, \ 0A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition 1.1.9. The $m \times n$ matrix whose entries are all zero is called the *zero* ($m \times n$) matrix.

1.2. **Matrix Multiplication.**

Matrix multiplication is very different from addition and subtraction.

In this subsection we describe how (and when) matrices can be multiplied together. This procedure can be complicated so we begin with examples before giving the general picture.

Example 1.2.1. A salesperson sells items of three types I, II and III, costing €10, €20 and €30, respectively. The next table shows how many items of each type are sold on Monday a.m. and p.m.

	Type I	Type II	Type III
morning	3	4	1
afternoon	5	2	2

Let A denote the matrix

$$\begin{pmatrix} 3 & 4 & 1 \\ 5 & 2 & 2 \end{pmatrix}$$

Let B denote the 3×1 matrix whose entries are the prices of items of Type I, II and III, respectively.

$$B = \begin{pmatrix} 10 \\ 20 \\ 30 \end{pmatrix}$$

(B is a column vector - see Remarks 1.1.3 (2)).

Let C denote the 2×1 matrix (another column vector) whose entries are the total income from morning sales and the total income from afternoon sales, respectively:

1st entry of C : $(3 \times 10) + (4 \times 20) + (1 \times 30) = 140$

2nd entry of C : $(5 \times 10) + (2 \times 20) + (2 \times 30) = 150$

I.e., $C = \begin{pmatrix} 140 \\ 150 \end{pmatrix}$.

How do the entries of C depend on those of A and B?

The 1st entry of C comes from combining the *first* row of A with the column of B:

product of 1st entries + product of 2nd entries + product of 3rd entries

$$(3 \times 10) + (4 \times 20) + (1 \times 30)$$

The 2nd entry of C comes from combining the *second* row of A with the column of B in the same way.

$$(5 \times 10) + (2 \times 20) + (2 \times 30)$$

This scheme for 'combining' rows of one matrix with columns of another leads to the definition of matrix multiplication. In this case C is the product 'A multiplied by B', i.e., $C = AB$.

Let A, B be $m \times p$ and $q \times n$ matrices, respectively. The *product* AB of A and B is defined only if $p = q$. In this case, AB is a $m \times n$ matrix.

Example 1.2.2. On slides.

Find AB when $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 2 \end{pmatrix}$.

Here, A is a 2×3 matrix and B is a 3×2 matrix, so AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 5 & 9 \\ 3 & -1 \end{pmatrix}$$

Definition 1.2.3. Let A, B be $m \times p$ and $q \times n$ matrices, respectively. The *product* AB of A and B is defined only if $p = q$.

In this case, AB is a $m \times n$ matrix, and the (i, j) entry $(AB)_{ij}$ of AB is

'the entries of the ith row of A combined with those of the jth column of B as in Example 1.2.2'.

Example 1.2.4. If A and B are as in Example 1.2.2 then BA is a 3×3 matrix. The $(2, 3)$ entry of BA, $(BA)_{23}$ is

'2nd row of B combined with 3rd column of A', i.e.

$$(BA)_{23} = 1 \times 3 + (-1) \times (-1) = 4.$$

The complete product is

$$BA = \begin{pmatrix} 7 & -3 & 8 \\ 1 & -1 & 4 \\ 2 & 0 & -2 \end{pmatrix}$$

Warning 1.2.5. Matrix multiplication is *not* commutative! i.e. in general, $AB \neq BA$. In Examples 1.2.2 and 1.2.4, we saw that AB and BA were both defined, but did not even have the same size. It is also possible for only one of AB and BA to be defined, e.g. if A is 2×4 and B is 4×3. Even if AB and BA are both defined and have the same size (e.g. if both are 3×3), the two products are typically different.

Remark 1.2.6 (General properties of matrix arithmetic)**.** In the following, A, B and C are matrices and for each item below we assume that their sizes are such that all indicated additions and multiplications are defined.

(1) $A + B = B + A$: matrix addition is *commutative.*
(2) $(A + B) + C = A + (B + C)$: matrix addition is *associative.*
(3) AB is typically not equal to BA, even when both are defined : matrix multiplication is *not* commutative.
(4) $(AB)C = A(BC)$: matrix multiplication is *associative.*
(5) Distributive laws for matrix multiplication over matrix addition:
$(A + B)C = AC + BC$
$A(B + C) = AB + AC$
(6) Whenever k is a real number, we have $(kA)B = A(kB) = k(AB)$ and $k(A + B) = kA + kB$.

Matrices share properties (1), (2), (4) and (5) with real numbers, but *not* (3)!

As with ordinary arithmetic of real numbers, matrix multiplication is done before addition or subtraction, e.g., $AB + C$ means $(AB) + C$, *not* $A(B + C)$.

Recall that a matrix is called *square* if it has the same number of rows as columns, i.e., if it is a $n \times n$ matrix for some n. If A and B are two 3×3 matrices then $A + B$, $A - B$ and the products AB and BA are all defined and all 3×3 matrices.

Fix n. Within the set of $n \times n$ matrices, we can add, subtract and multiply any matrix by any other, and stay inside the set of $n \times n$ matrices.

Definition 1.2.7. The set of $n \times n$ matrices whose entries are real numbers is denoted $M_n(\mathbb{R})$. If A is a $n \times n$ matrix we can write $A \in M_n(\mathbb{R})$.

Definition 1.2.8 (Matrix powers)**.** Let $A \in M_n(\mathbb{R})$, i.e., A is a $n \times n$ matrix, and let k be a positive integer. The kth power A^k of A is the $n \times n$ matrix

$$\underbrace{A \times A \times \cdots \times A}_{k \text{ times}}$$

Example 1.2.9. In $M_2(\mathbb{R})$, if $A = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$, then

$$A^2 = AA = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix} \quad \text{and}$$

$$A^3 = AAA = A^2A = \begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -13 \\ 0 & 27 \end{pmatrix}.$$

Warning 1.2.10. Recall that if x is a number and $x^2 = 0$ then x has to be 0. However, if A is a square matrix and A^2 is the 2×2 zero matrix, then A *does not* have to be the zero matrix. E.g., if

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition 1.2.11 (Transposes)**.** If A is a $m \times n$ matrix then the *transpose* A^T of A is the $n \times m$ matrix whose entries are given by

$$(A^T)_{ij} = a_{ji}.$$

In practice, this means that the ith row of A^T is the ith column of A.

Example 1.2.12. If $A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 3 & 7 \end{pmatrix}$ then

$$A^T = \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ 5 & 7 \end{pmatrix}.$$

Transposes are taken before matrix addition, matrix multiplication and scalar multipliction:

- $A + B^T$ means $A + (B^T)$, *not* $(A + B)^T$.
- AB^T means $A(B^T)$, *not* $(AB)^T$.

Some useful properties of the transpose are:

(1) $(A^T)^T = A$.
(2) If $A + B$ can be formed then $(A + B)^T = A^T + B^T$.
(3) If AB can be formed then $(AB)^T = B^T A^T$.
(4) If c is a number then $(cA)^T = c(A^T)$.

1.3. Matrix Inverses.

If we divide a real number by 5 we are multiplying it by $\frac{1}{5}$. That is, $\frac{1}{5}$ is the *reciprocal* or *multiplicative inverse* of 5 in $\mathbb{R}$. This means

$$\tfrac{1}{5} \times 5 = 1,$$

i.e., if you multiply 5 by $\frac{1}{5}$, you get 1; multiplying by $\frac{1}{5}$ 'reverses' the work of multiplying by 5.

More generally, if a is a non-zero real number then $\frac{1}{a}$ is the inverse of a. When given an equation like $ax = b$, where a and b are known to us, we solve it by multiplying both sides by the inverse of a:

$$x = 1x = \tfrac{1}{a}ax = \tfrac{1}{a}b = \tfrac{1b}{a} = \tfrac{b}{a}$$

E.g. if $5x = 4$ then $x = \frac{4}{5}$.

Our aim now is to introduce a similar procedure to solve *matrix equations* like $AX = B$, and much more besides. To do this, we need the concept of a matrix inverse.

However, before we can do this, we need to find a matrix which behaves something like the number 1. The number 1 is special because when you multiply any number by 1, you change nothing (and it is the only number with the property).

Example 1.3.1. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ and let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Find AI and IA.

$$\begin{aligned} AI &= \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 1 + 3 \times 0 & 2 \times 0 + 3 \times 1 \\ (-1) \times 1 + 2 \times 0 & (-1) \times 0 + 2 \times 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \\ &= A \end{aligned}$$

$$\begin{aligned} IA &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 2 + 0 \times (-1) & 1 \times 3 + 0 \times 2 \\ 0 \times 2 + 1 \times (-1) & 0 \times 3 + 1 \times 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \\ &= A \end{aligned}$$

Both AI and IA are equal to A: multiplying A by I (on the left or right) does not affect A.

In general, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *any* 2×2 matrix, then

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

and $IA = A$ also.

Definition 1.3.2. $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 *identity matrix* (it's sometimes denoted I_2).

Remark 1.3.3.

(1) I_2 behaves in $M_2(\mathbb{R})$ like the real number 1 behaves in $\mathbb{R}$ - multiplying a real number x by 1 has no effect on x.

(2) The 3×3 identity matrix is $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Check that if A is any 3×3 matrix then $AI_3 = I_3 A = A$.

(3) For any positive integer n, the $n \times n$ *identity matrix* I_n is defined by

$$I_n = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{pmatrix}$$

(I_n has 1s along the 'main diagonal' and 0s elsewhere). The entries of I_n are given by:

$$(I_n)_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 1.3.4. *If A is any $n \times n$ matrix then $AI_n = I_n A = A$. I.e., multiplying A on the left or right by I_n leaves A unchanged.*

Now we have the $n \times n$ identity matrices, we can ask which $n \times n$ matrices have multiplicative inverses.

Example 1.3.5. Let $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and let $B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$.

Then

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$BA = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

So B is an inverse for A.

In the same way that $\frac{1}{5}$ 'reverses' the effect of 5, by getting us back to 1 ($\frac{1}{5} \times 5 = 1$), so B reverses the effect of A by getting us back to I_2 ($AB = BA = I_2$).

Definition 1.3.6. Let A be a $n \times n$ matrix. If B is a $n \times n$ matrix for which

$$AB = I_n \quad \text{and} \quad BA = I_n$$

then B is called an *inverse* for A and we say that A is *invertible*.

Remark 1.3.7.

(1) Suppose B and C are *both* inverses for a particular matrix A, i.e.

$$BA = AB = I_n \quad \text{and} \quad CA = AC = I_n.$$

Then

$$\begin{aligned} (BA)C &= I_n C = C \\ \text{and also} \quad (BA)C &= B(AC) = BI_n = B \end{aligned}$$

Hence $B = C$, and if A has an inverse, its inverse is *unique*. Thus we can talk about *the* inverse of a matrix.

(2) Not every square matrix has an inverse. For example the 2×2 zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not.

(3) The inverse of a $n \times n$ matrix A, if it exists, is denoted A^{-1}.

Some useful properties of the inverse are:

(1) If A is invertible then $(A^{-1})^{-1} = A$.
(2) If AB can be formed and is invertible then $(AB)^{-1} = B^{-1}A^{-1}$.
(3) If c is a non-zero number and A is invertible then $(cA)^{-1} = \frac{1}{c}A^{-1}$.
(4) If A is invertible then $(A^T)^{-1} = (A^{-1})^T$.

Given a square matrix A, how do we

(a) decide if A^{-1} exists, and
(b) if so, work out what it is?

In the 2×2 case, there is a nice formula to work out matrix inverses.

Example 1.3.8. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, and let us suppose that

$$ad - bc \neq 0.$$

(For instance, in Example 1.3.5 we had $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$, and in this case $ad - bc = 2 \times 3 - 1 \times 5 = 6 - 5 = 1 \neq 0$.)

Now consider

$$B = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

We calculate the product AB:

$$
\begin{aligned}
AB &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
&= \frac{1}{ad-bc} \begin{pmatrix} ad+b(-c) & a(-b)+ba \\ cd+d(-c) & c(-b)+da \end{pmatrix} \\
&= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2
\end{aligned}
$$

Similarly (and you should check this) $BA = I_2$, so B is the *inverse* of A, i.e., $B = A^{-1}$.

Check that if you apply the formula to A in Example 1.3.5 then you get the inverse.

However, what if A is as above, but $ad - bc = 0$? It turns out that, in this case, A does *not* have an inverse. We will return to this number $ad - bc$ in Section 1.6

Example 1.3.9. The matrix

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

does not have an inverse.

To show this, we assume that A *does* have an inverse

$$A^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

and then reach a conclusion which doesn't make any sense.

Take a new matrix $B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. If we compute the product AB, we get

$$AB = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 2 \times (-1) \\ (-2) \times 2 + (-4) \times (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So if we multiply AB by A^{-1} on the left, we get

$$A^{-1}(AB) = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p \times 0 + q \times 0 \\ r \times 0 + s \times 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

regardless of whatever A^{-1} is supposed to be. On the other hand, we can calculate the same product in a different way by using the rules of inverses

$$A^{-1}(AB) = (A^{-1}A)B = I_2 B = B.$$

Putting these together gives

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = B = A^{-1}(AB) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which doesn't make any sense.

The only way to get out of this is to say that our original assumption about A^{-1} is wrong: A does *not* have an inverse.

What about 3×3 matrices, or 4×4, 27×27? Even in the 3×3 case, while there is a formula, it isn't very nice. In larger cases the formulae become completely impossible to use, so we have to find another way of determining inverses. See Section 1.5.

1.4. **Systems of Linear Equations.**

Consider the equation

$$2x + y = 3.$$

This is a *linear equation* in the variables x and y. As it stands, the statement '$2x + y = 3$' is neither true nor false: it is just a statement involving the abstract variables x and y. However if we replace x and y with some particular pair of real numbers, the statement will become either true or false. E.g.

(1) Putting $x = 1,\ y = 1$ gives $2x + y = 2 \times 1 + 1 = 3$: True
(2) Putting $x = 1,\ y = 2$ gives $2x + y = 2 \times 1 + 2 \neq 3$: False

Definition 1.4.1. A pair (x_0, y_0) of real numbers is a *solution* to the equation $2x + y = 3$ if setting $x = x_0$ and $y = y_0$ makes the equation true; i.e. if $2x_0 + y_0 = 3$.

E.g. $(1, 1)$ and $(0, 3)$ are solutions, but $(1, 4)$ is not a solution since setting $x = 1,\ y = 4$ gives $2x + y = 2 \times 1 + 4 \neq 3$.

The set of all solutions to the equation is called its *solution set*. Given several linear equations, it is sometimes necessary to look for solutions which solve all of them simultaneously.

Example 1.4.2. On slides.
Suppose you need to take taxis to various places around town. Taxi company A has a €4 initial charge followed by €1/mile, while company B has a €3 initial charge followed by €1.10/mile. So when should you choose A over B?

Solution

Company A's price is given by $y = 4 + x$ and company B's is $y = 3 + 1.1x$, where x is the number of miles.

Recall that the 2-dimensional *Cartesian plane* is described by a pair of perpendicular axes, labelled x and y. A point is described by a pair of real numbers, its x and y-coordinates.

The *solution sets* of both linear equations can be represented by lines in the plane. See slides.

Where do the lines cross? We use algebra to solve for x:

$$\begin{aligned} x &= y-4 \qquad \text{(A)} \qquad (\text{or } x-y=-4) \\ 1.1x &= y-3 \qquad \text{(B)} \qquad (\text{or } 1.1x-y=-3) \end{aligned}$$

Subtracting (A) from (B) gives

$$0.1x = (y-3)-(y-4) = 1 \qquad \text{(C)} \qquad \text{we have } \textit{eliminated } y$$

and multiplying (C) by 10 gives $x = 10$. Substituting $x = 10$ back into $y = 4 + x$ gives $y = 14$. So $(10, 14)$ is a solution (the only solution) to *both (A) and (B)*.

This corresponds to the fact that the point $(10, 14)$ is the intersection of both lines: the point $(10, 14)$ lives in *both* solution sets.

Now we can say that for journeys of under 10 miles, company B is cheaper, while company A is cheaper for journeys of more than 10 miles.

This kind of 'ad hoc' approach may not always work if we have a more complicated system, involving a greater number of variables, or more equations. We will devise a general strategy for solving complicated systems of linear equations.

Example 1.4.3. On slides.
Find all solutions of the following system:

$$\begin{array}{rcrcrcl} x &+& 2y &-& z &=& 5 \\ 3x &+& y &-& 2z &=& 9 \\ -x &+& 4y &+& 2z &=& 0 \end{array}$$

In other (perhaps simpler) systems like Example 1.4.2 we were able to find solutions by simplifying the system (perhaps by eliminating certain variables) through operations such as:

(1) Adding/subtracting one equation to/from another (e.g. in the hope of eliminating a variable).
(2) Multiplying one equation by a non-zero constant.

A similar approach will work for Example 1.4.3 but with this and other harder examples it may not always be clear how to proceed. We now develop a new technique, both for describing our system and for applying operations of the above types more systematically and with greater clarity.

Back to Example 1.4.3. We associate a matrix to our system.

$$\begin{array}{rcrcrcl} x &+& 2y &-& z &=& 5 \\ 3x &+& y &-& 2z &=& 9 \\ -x &+& 4y &+& 2z &=& 0 \end{array}$$

$$\begin{pmatrix} 1 & 2 & -1 & 5 \\ 3 & 1 & -2 & 9 \\ -1 & 4 & 2 & 0 \end{pmatrix} \begin{array}{l} \text{Eqn 1} \\ \text{Eqn 2} \\ \text{Eqn 3} \end{array}$$

(1) Row 1 of this matrix comprises first the coefficients of x, y, z and then the number on the right hand side of Eqn 1.
(2) Similarly for rows 2 and 3.
(3) The columns correspond (from left to right) first to the variables x, y, z and then the right hand side of the system as a whole.

Definition 1.4.4. The above matrix is called the *augmented matrix* of the system of equations in Example 1.4.3.

In solving systems of equations we are allowed to perform operations of the following types:

(1) Multiply an equation by a non-zero constant.
(2) Add one equation (or a non-zero constant multiple of one equation) to another equation.

These correspond to the following operations on the augmented matrix:

(1) Multiply a *row* by a non-zero constant.
(2) Add a multiple of one row to another row.
(3) We also allow operations of the following type: interchange two rows in the matrix (this only amounts to writing down the equations of the system in a different order).

Definition 1.4.5. Operations on a matrix of these 3 types are called *Elementary Row Operations* (EROs).

The EROs are on the slides. See how the corresponding system of equations changes and gets simpler. The whole point of using EROs is that the system of equations gets simpler, but the *solutions of the system are preserved.* So the solutions of the system at the end will be the same as the solutions of the original system.

Step 1

$$\begin{array}{c|rrrr} R3 & -1 & 4 & 2 & 0 \\ + & & & & \\ R1 & 1 & 2 & -1 & 5 \\ \hline \text{new } R3 & 0 & 6 & 1 & 5 \end{array}$$

Step 2

$$\begin{array}{c|rrrr} R2 & 3 & 1 & -2 & 9 \\ - & & & & \\ 3R1 & 3 & 6 & -3 & 15 \\ \hline \text{new } R2 & 0 & -5 & 1 & -6 \end{array}$$

Step 3

$$\begin{array}{c|rrrr} R2 & 0 & -5 & 1 & -6 \\ + & & & & \\ R3 & 0 & 6 & 1 & 5 \\ \hline \text{new } R2 & 0 & 1 & 2 & -1 \end{array}$$

Step 4

$R3$	0	6	1	5
$-$				
$6R2$	0	6	12	-6
new $R3$	0	0	-11	11

Step 5

$R3$	0	0	-11	11
$-\frac{1}{11}R3$	0	0	1	-1

We have produced a new, simpler, system of equations.

$$\begin{array}{rcrcrcrl} x & + & 2y & - & z & = & 5 & (A) \\ & & y & + & 2z & = & -1 & (B) \\ & & & & z & = & -1 & (C) \end{array}$$

This is easily solved.

Backsubstitution $\left\{ \begin{array}{lll} (C) & z = -1 & \\ (B) & y = -1 - 2z & \Longrightarrow \; y = -1 - 2(-1) = 1 \\ (A) & x = 5 - 2y + z & \Longrightarrow \; x = 5 - 2(1) + (-1) = 2 \end{array} \right.$

So the solution is $(2, 1, -1)$ or $x = 2,\ y = 1, z = -1$. You should check that this is a solution of the original system. It is the only solution both of the final system and of the original one (and every intermediate one).

The matrix obtained in Step 5 above is in *row-echelon form.*

Definition 1.4.6. A matrix A is in row-echelon form (REF) if

(1) The first non-zero entry in each row is a 1 (called a *leading 1*).
(2) If a column contains a leading 1, then every entry of the column below the leading 1 is a zero.
(3) As we move downwards through the rows of the matrix, the leading 1s move from left to right.
(4) Any rows consisting entirely of 0s are grouped together at the bottom of the matrix.

General Strategy to Obtain a Row-Echelon Form

(1) Get a 1 as the top left entry of the matrix.
(2) Use this first leading 1 to 'clear out' the rest of the first column, by adding suitable multiples of row 1 to subsequent rows.
(3) If column 2 contains non-zero entries (other than in the first row), use EROs to get a 1 as the second entry of row 2. After this step the matrix will look

like the following (where the entries represented by stars can be anything):

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ 0 & * & \dots & \dots & \dots \\ \vdots & \vdots & & & \vdots \\ 0 & * & \dots & \dots & \dots \end{pmatrix}$$

(4) Now use this second leading 1 to 'clear out' the rest of column 2 (below row 2) by adding suitable multiples of row 2 to subsequent rows. After this step the matrix will look like the following:

$$\begin{pmatrix} 1 & * & * & \dots & \dots \\ 0 & 1 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ 0 & 0 & * & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \dots & \dots \end{pmatrix}$$

(5) Now go to column 3. If it has non-zero entries (other than in the first two rows) get a 1 as the third entry of row 3. Use this third leading 1 to clear out the rest of column 3, then proceed to column 4. Continue until a row-echelon form is obtained.

Example 1.4.7. On slides.
Let A be the matrix

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$$

Reduce A to row-echelon form.

Solution

(1) Get a 1 as the first entry of row 1. Done.
(2) Use this first leading 1 to clear out column 1 as follows:

$$\begin{array}{rcl} R2 & \to & R2 - 2R1 \\ R3 & \to & R3 - R1 \end{array} \quad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 3 & 1 & -2 & 8 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$

(3) Get a leading 1 as the second entry of row 2, for example as follows:

$$R2 \to R2 + R3 \quad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & -2 & 3 & 5 & 2 \end{pmatrix}$$

(4) Use this leading 1 to clear out whatever appears below it in column 2:

$$R3 \rightarrow R3 + 2R2 \qquad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 11 & 11 & 22 \end{pmatrix}$$

(5) Get a leading 1 in row 3:

$$R3 \rightarrow \tfrac{1}{11} \times R3 \qquad \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

This matrix is now in row-echelon form.

Definition 1.4.8. A matrix is in *reduced row-echelon form (RREF)* if

(1) It is in row-echelon form, and
(2) If a particular column contains a leading 1, then *all* other entries of that column are 0s.

If we have a REF, we can use more EROs to obtain a RREF (using EROs to obtain a RREF is called *Gaussian elimination* or *Gauss-Jordan elimination*).

Example 1.4.3 once again. See accompanying slides.

Step 6

$$\begin{array}{c|cccc} R1 & 1 & 2 & -1 & 5 \\ - & & & & \\ 2R2 & 0 & 2 & 4 & -2 \\ \hline \text{new } R1 & 1 & 0 & -5 & 7 \end{array}$$

Step 7

$$\begin{array}{c|cccc} R2 & 0 & 1 & 2 & -1 \\ - & & & & \\ 2R3 & 0 & 0 & 2 & -2 \\ \hline \text{new } R2 & 0 & 1 & 0 & 1 \end{array}$$

Step 8

$$\begin{array}{c|cccc} R1 & 1 & 0 & -5 & 7 \\ + & & & & \\ 5R3 & 0 & 0 & 5 & -5 \\ \hline \text{new } R1 & 1 & 0 & 0 & 2 \end{array}$$

The matrix is now in *reduced* row-echelon form (see step 8 on slides).

The technique outlined in this example will work in general to obtain a RREF from a REF.

Example 1.4.9. On slides.
In Example 1.4.7 we obtained the following REF:

$$\begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix} \qquad \text{(REF, not } \textit{reduced} \text{ REF)}$$

To get the RREF from this REF:

(1) Look for the leading 1 in row 2 - it is in column 2. Eliminate the non-zero entry *above* this leading 1 by adding a suitable multiple of row 2 to row 1.

$$R1 \rightarrow R1 + R2 \qquad \begin{pmatrix} 1 & 0 & 3 & 5 & 10 \\ 0 & 1 & 4 & 3 & 10 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

(2) Look for the leading 1 in row 3 - it is in column 3. Eliminate the non-zero entries *above* this leading 1 by adding suitable multiples of row 3 to rows 1 and 2.

$$\begin{matrix} R1 & \rightarrow & R1 - 3R3 \\ R2 & \rightarrow & R2 - 4R3 \end{matrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

This matrix is in *reduced* row-echelon form.

The technique outlined in this example will work in general to obtain a RREF from a REF.

Leading Variables and Free Variables

In Examples 1.4.2 and 1.4.3, we obtained *unique* solutions of the systems, i.e., in both cases there is only one combination of x and y (in Example 1.4.2) or x, y and z (in Example 1.4.3) which solves the system. However, often there can be more than one solution to a given system. We deal with this situation here.

Example 1.4.10. On slides.
Find the general solution of the following system:

$$\begin{array}{rcrcrcrcrr} x_1 & - & x_2 & - & x_3 & + & 2x_4 & = & 0 & \text{I} \\ 2x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & 8 & \text{II} \\ x_1 & - & 3x_2 & + & 2x_3 & + & 7x_4 & = & 2 & \text{III} \end{array}$$

Solution:

(1) Write down the augmented matrix of the system:

$$\begin{matrix} \text{Eqn I} \\ \text{Eqn II} \\ \text{Eqn III} \end{matrix} \begin{pmatrix} 1 & -1 & -1 & 2 & 0 \\ 2 & 1 & -1 & 2 & 8 \\ 1 & -3 & 2 & 7 & 2 \end{pmatrix}$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

This is the matrix of Example 1.4.7.

(2) Use Gauss-Jordan elimination to find a RREF from this augmented matrix. This is the matrix of Example 1.4.9. See slide as well.

$$\text{RREF}: \quad \begin{pmatrix} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$
$$\quad x_1 \quad x_2 \quad x_3 \quad x_4$$

This matrix corresponds to a new system of equations:

$$\begin{array}{ccccccc} x_1 & + & 2x_4 & = & 4 & & (A) \\ x_2 & - & x_4 & = & 2 & & (B) \\ x_3 & + & x_4 & = & 2 & & (C) \end{array}$$

The RREF involves 3 leading 1s, one in each of the columns corresponding to the variables x_1, x_2 and x_3. The column corresponding to x_4 contains no leading 1. We make a distinction between these cases.

Definition 1.4.11. The variables whose columns in the RREF contain leading 1s are called *leading variables*. A variable whose column in the RREF does not contain a leading 1 is called a *free variable*.

So in this example the leading variables are x_1, x_2 and x_3, and the variable x_4 is free. What does this distinction mean? We rewrite the system corresponding to the RREF:

$$\begin{array}{ccccccc} x_1 & = & 4 & - & 2x_4 & & (A) \\ x_2 & = & 2 & + & x_4 & & (B) \\ x_3 & = & 2 & - & x_4 & & (C) \end{array}$$

i.e. this RREF tells us how the values of the leading variables x_1, x_2 and x_3 *depend* on that of the free variable x_4 in a solution of the system. In a solution, the free variable x_4 can be *any* real number. However, once a value for x_4 is chosen, values are immediately assigned to x_1, x_2 and x_3 by equations A, B and C above. For example

(a) Choosing $x_4 = 0$ gives $x_1 = 4 - 2(0) = 4,\ x_2 = 2 + (0) = 2,\ x_3 = 2 - (0) = 2$. Check that $x_1 = 4,\ x_2 = 2,\ x_3 = 2,\ x_4 = 0$ is a solution of the (original) system.

(b) Choosing $x_4 = 3$ gives $x_1 = 4 - 2(3) = -2,\ x_2 = 2 + (3) = 5,\ x_3 = 2 - (3) = -1$. Check that $x_1 = -2,\ x_2 = 5,\ x_3 = -1,\ x_4 = 3$ is a solution of the (original) system.

Different choices of value for x_4 will give different solutions of the system. The number of solutions is *infinite*.

The *general solution* is usually described by the following type of notation. We assign the *parameter* name t to the value of the variable x_4 in a solution (so t may assume any real number as its value). We then have

$$x_1 = 4 - 2t,\ x_2 = 2 + t,\ x_3 = 2 - t,\ x_4 = t;\ t \in \mathbb{R}$$

or

$$(x_1, x_2, x_3, x_4) = (4 - 2t, 2 + t, 2 - t, t);\ t \in \mathbb{R}$$

This general solution describes the infinitely many solutions of the system. We get a *particular* solution by choosing a specific numerical value for t: this determines specific values for x_1, x_2, x_3 and x_4.

Consistent and Inconsistent Systems

It is also possible that a given system can have no solutions at all.

Example 1.4.12. On slides.

The system of equations corresponding to this REF has as its third equation

$$0x + 0y + 0z = 1 \quad \text{i.e.} \quad 0 = 1$$

This equation clearly has no solutions - no assignment of numerical values to x, y and z will make the value of the expression $0x + 0y + 0z$ equal to anything but 0. Hence the system has no solutions.

Definition 1.4.13. A system of linear equations is called *inconsistent* if it has no solutions. A system which has at least one solution is called *consistent*.

If a system is inconsistent, a REF obtained from its augmented matrix will include a row of the form 0 0 0 ... 0 1, i.e. it will have a leading 1 in its rightmost column. Such a row corresponds to an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = 1$, which certainly has no solution.

Solving Systems of Linear Equations: Summary

Given any system of linear equations there are three possible outcomes.

(1) **Unique solution**: this will happen if the system is consistent and all variables are leading variables. In this case the RREF obtained from the augmented matrix has the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & * \\ 0 & 1 & 0 & \dots & 0 & * \\ 0 & 0 & 1 & \dots & 0 & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & * \end{pmatrix}$$

with possibly some additional rows consisting entirely of 0s at the bottom. The unique solution can be read from the rightmost column. See the RREF from Example 1.4.3.

(2) **Infinitely many solutions**: this happens if the system is consistent but at least one of the variables is free. In this case the system has a *general solution* with at least one parameter. See Examples 1.4.7, 1.4.9 and 1.4.10.

(3) **No Solutions**: the system may be inconsistent, i.e., it has no solutions. This happens if a REF obtained from the augmented matrix has a leading 1

in its rightmost column.

$$\begin{pmatrix} & & \vdots & & & \\ & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

See Example 1.4.12.

1.5. **Inverting matrices again.**

In section 1.3 we introduced matrix inverses. In the 2×2 case, there is a convenient formula to work out inverses - see Example 1.3.8. However, for matrices of greater size: 3×3, 17×17, etc, the available formulae are difficult and often impossible to use. Instead, EROs can be used in a much more efficient way to work out inverses. Rather than use a formula we use a procedure.

Example 1.5.1. Let A be a 3×3 matrix e.g. $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 2 \end{pmatrix}$

Find A^{-1}.

Solution We are looking for a 3×3 matrix A^{-1} with $A \times A^{-1} = I_3$. Recall that I_3 is the 3×3 *identity matrix*. A^{-1} (if it exists) can be found using EROs as follows:

(1) Write down a 3×6 matrix B, whose first 3 columns comprise A and whose second 3 columns comprise I_3:

$$B = \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

(2) Use elementary row operations to obtain a RREF from B:

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{c} R2 \to R2 - 2R1 \\ R3 \to R3 - R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$R3 \leftrightarrow R2 \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -6 & -3 & -2 & 1 & 0 \end{array}\right)$$

$$\begin{array}{c} R1 \to R1 - 3R2 \\ \\ R3 \to R3 + 6R2 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 4 & 0 & -3 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 3 & -8 & 1 & 6 \end{array}\right)$$

$$R3 \to \tfrac{1}{3}R3 \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 4 & 0 & -3 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{3} & 2 \end{array}\right)$$

$$\begin{array}{l} R1 \to R1 + 2R3 \\ R2 \to R2 - R3 \end{array} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{3} & \frac{2}{3} & 1 \\ 0 & 1 & 0 & \frac{5}{3} & -\frac{1}{3} & -1 \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{3} & 2 \end{array}\right)$$

(3) In this RREF, each of the first 3 columns contains a leading 1, and the first 3 columns comprise the 3×3 identity matrix I_3.
The 3×3 matrix consisting of the last three columns is the inverse of A

$$A^{-1} = \begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & 1 \\ \frac{5}{3} & -\frac{1}{3} & -1 \\ -\frac{8}{3} & \frac{1}{3} & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 & 2 & 3 \\ 5 & -1 & -3 \\ -8 & 1 & 6 \end{pmatrix}$$

Procedure To invert a $n \times n$ matrix A using EROs

(1) Form the $n \times 2n$ matrix $B = (A|I_n)$
(2) Use elementary row operations to reduce B to RREF.
(3) If each of the first n columns of the RREF contains a leading 1, then the left hand side of the RREF is I_n, and the matrix formed by the last n columns of the RREF is A^{-1}, the inverse of A.
(4) If not all of the 1st n columns of the RREF of B contain leading 1s, then A is *not* invertible, i.e., does not have an inverse.

Example 1.5.2. Does $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$ have an inverse?

Solution

(1) Form the 3×6 matrix $B = \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$.
(2) Use EROs to reduce B to RREF:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 4 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 - R1 \qquad \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} R1 \to R1 - 2R2 \\ \\ R3 \to R3 - 2R2 \end{array} \qquad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 1 \end{array}\right)$$

$$R3 \to -R3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right)$$

$$R1 \to R1 - R3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & -4 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right)$$

In this RREF, the 3rd column does *not* contain a leading 1, so A does *not* have an inverse.

1.6. Determinants.

While the above method can tell us if a matrix has an inverse, there is a quicker method that we can use to see if a given matrix has an inverse. We can then use the above method to find the inverse if it exists. We noted after Example 1.3.8 that a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has an inverse if and only if $ad - bc \neq 0$. The number $ad - bc$ is called the **determinant** of the matrix A (it determines if the matrix has an inverse). The determinant of A can be denoted by $\det(A)$ or $|A|$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

Luckily there is a method for calculating the determinant of a square matrix of any size and it is still true that a square matrix is invertible if and only if its determinant is non-zero. The idea behind calculating determinants is to express the determinant of a $n \times n$ matrix in terms of determinants of $(n-1) \times (n-1)$ matrices which are derived from the $n \times n$ matrix. We will first look at the determinant of a 3×3 matrix since this shows all the essential features of the method.

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Notice that the 2×2 determinant multiplying a_{11} is obtained from the original 3×3 matrix by cancelling out the row and the column containing a_{11} and similarly for the other two 2×2 determinants.

To demonstrate this method, we will look again at Example 1.5.2.

Example 1.6.1. Does $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$ have an inverse?

Solution This time we will calculate the determinant of $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$.

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix}$$

$$
\begin{aligned}
&= 1(1\times 1 - 1\times 4) - 2(0\times 1 - 1\times 1) - 1(0\times 4 - 1\times 1)\\
&= 1(-3) - 2(-1) - 1(-1)\\
&= -3 + 2 + 1\\
&= 0.
\end{aligned}
$$

Since the determinant of A is zero, we again obtain that A is not invertible.

Note that this gives a quicker solution than the one we obtained in Example 1.5.2, so when faced with finding the inverse of a matrix, a good strategy can be to calculate the determinant to see if the inverse exists and then use row reduction if it does. As we noted above there is another method for finding the inverse of a matrix but it involves a lot more computation, especially for large matrices, and we will not cover it in this course.

The determinant of a square matrix of any size can be calculated as follows.

Definition 1.6.2 (Cofactor)**.** Let $A = (a_{ij})$ be an $n \times n$ matrix. The *cofactor* A_{ij} associated with the entry a_{ij} is defined to be $A_{ij} = (-1)^{i+j} \det(\mathbf{A_{ij}})$, where $\mathbf{A_{ij}}$ is the $(n-1)\times(n-1)$ sub-matrix of A obtained by deleting the ith row and jth column from A (i.e., the row and column containing a_{ij}).

Using this definition, we have that the determinant of a $n \times n$ matrix is given by

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n}\\
a_{21} & a_{22} & \cdots & a_{2n}\\
\vdots & \vdots & & \vdots\\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
= a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.
$$

Remark 1.6.3. Here we have 'expanded' the determinant along the top row. There is nothing special about the top row: we could equally well have expanded the determinant along any particular row of column. This can be useful if it turns out that there are a lot of zeros in a row or column. If we expand along this row or column then it cuts down on the amount of calculations. If you decide to do this then the sign of $(-1)^{i+j}$ can be remembered as
$$
\begin{vmatrix}
+ & - & + & - & \cdots\\
- & + & - & + & \cdots\\
+ & - & + & - & \cdots\\
- & + & - & + & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix}.
$$

Some useful properties of the determinant are:

(1) If A has an entire row (or column) of zeros or if A has two equal rows (or columns) or if A has two rows (or columns) that are multiples of each other, then $\det(A) = 0$.
(2) If we swap two rows of a matrix then the sign of its determinant changes (but its absolute value remains the same).
(3) If A and B are two square matrices of the same size, then $\det(AB) = (\det(A))(\det(B))$.

(4) If I is the identity matrix (of any size), then $\det(I) = 1$.
(5) $\det(A^T) = \det(A)$.
(6) $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

1.7. Vectors.

We mentioned above that $1 \times n$ matrices are also called row vectors and $n \times 1$ matrices are also known as column vectors. When $n = 2$ or $n = 3$ we can represent vectors as directed lines in $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. For example in $\mathbb{R}^2$ the vector $\begin{pmatrix} 2 & 1 \end{pmatrix}$ (or equivalently the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$) can be represented as any of the lines in Figure 1. Note that sometimes row vectors are written with commas, i.e., as $(2, 1)$.

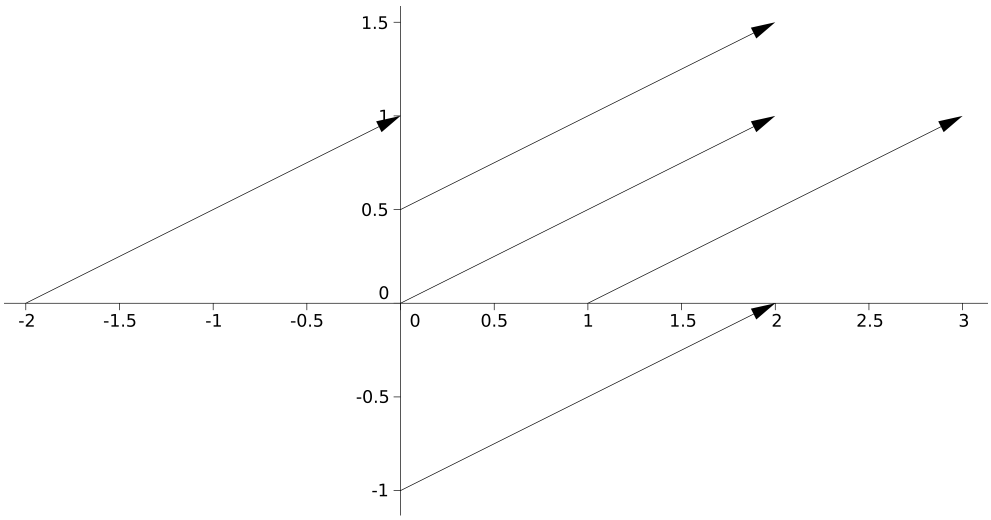

FIGURE 1. Representations of the vector (2 1).

Also note, since vectors are matrices, we have already defined how to add vectors and how to multiply them by numbers but it is more usual when writing vectors to write them as lower case letters which are either underlined or written with an arrow across the top, for example $\underline{v}$ or $\vec{v}$. In print vectors are also often written in bold face, i.e., as $\mathbf{v}$. You may also be familiar with the notation $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ which represent vectors of length one in the directions of the x, y and z axes respectively. Thus we can also write $(1, 2, 3) = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ for example. In $\mathbb{R}^2$ or $\mathbb{R}^3$ it is also easy to calculate the length of a vector and this measure can be extended to spaces of higher dimensions, where it is usually called the norm of a vector.

Definition 1.7.1 (Norm (length) of a vector)**.** Let $A = (a_{1j})$ be a $1 \times n$ vector. Then the *norm* (*length*) of A is given by

$$\|A\| = \sqrt{\sum_{j=1}^{n} (a_{1j})^2}.$$

That is to find the norm of a vector, we square all the components, add them up and then take the square root of the resulting number. Note that the norm of a vector is a non-negative number, as we would expect if it is measuring length. Of course we can make a similar definition for a column vector or for a vector written as $\mathbf{v} = (\mathbf{v_j})$. In more advanced mathematics we can also define the norm of a matrix but there the situation is a lot more complex since there are several different norms we can use and some of them are very difficult to calculate.

Example 1.7.2. Find the length of the vector $(-3, 2, 1)$.

Solution The length of $(-3, 2, 1)$ is given by

$$\|(-3, 2, 1)\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}.$$

Remark 1.7.3. If we want to find a unit vector (i.e., a vector of length one) in the direction of a given vector, then all we have to do is to divide each of the vector's components by the length of the vector.

Example 1.7.4. Find a unit vector in the same direction as the vector $(-3, 2, 1)$.

Solution By Example 1.7.2, the length of $(-3, 2, 1)$ is $\sqrt{14}$. Hence a unit vector in the direction of $(-3, 2, 1)$ is $(-3/\sqrt{14}, 2/\sqrt{14}, 1/\sqrt{14})$.

However there are a couple of important operations defined for vectors that are not usually defined for general matrices. The first of these is called the dot (or scalar) product and it allows us to combine two vectors of the same size to obtain a scalar (i.e., a number). We will define it using row vectors but as above, it could be equally well be defined using column vectors or vectors written as $\mathbf{v} = (v_j)$.

Definition 1.7.5 (Dot Product)**.** Let $A = (a_{1j})$ and $B = (b_{1j})$ be $1 \times n$ row vectors. Then the *dot product* of A with B is defined to be

$$A \cdot B = \sum_{j=1}^{n} a_{1j} b_{1j}.$$

That is to find the dot product of two vectors you multiply the first components, the second components and so on up to the nth components and then add them all up. Note, in contrast to the norm, the dot product of two vectors can result in a negative number.

Example 1.7.6. Find the dot product of $(1, 2, -3, 4)$ with $(-2, 3, 0, 5)$.

Solution

$$(1, 2, -3, 4) \cdot (-2, 3, 0, 5) = (1)(-2) + (2)(3) + (-3)(0) + (4)(5) = -2 + 6 + 0 + 20 = 24.$$

In $\mathbb{R}^2$ and $\mathbb{R}^3$ there is also a very useful geometric description of the dot product which enables us to find the angle between any two vectors. If θ is the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$ (see Figure 2) then $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \cos(\theta)$, where the dot on the left side of this equation signifies the dot product, while the dot on the right side of the equation signifies ordinary multiplication between two numbers.

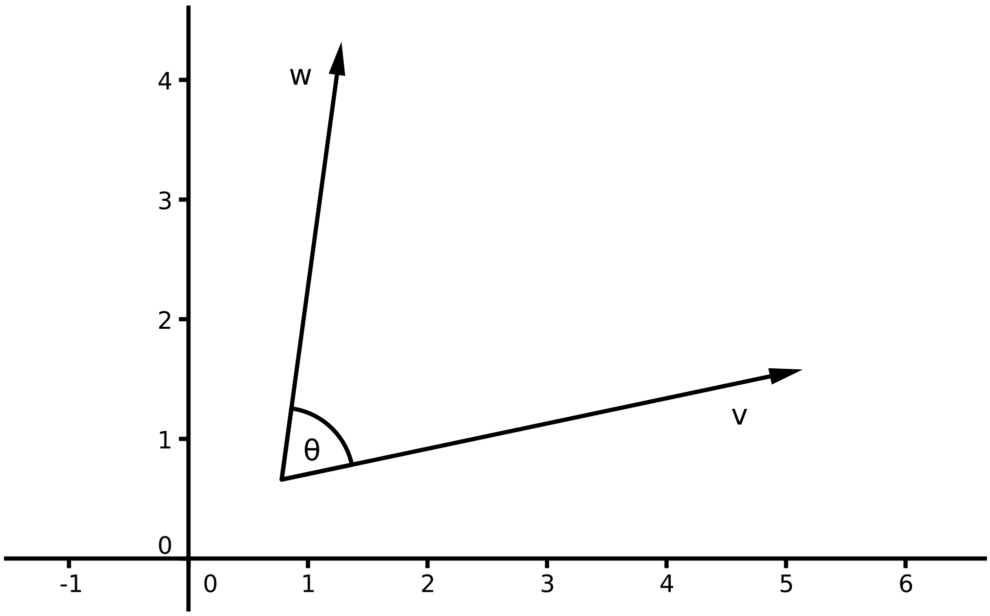

FIGURE 2. Angle between two vectors.

Thus if we have two vectors $\mathbf{v}$ and $\mathbf{w}$ which have an angle θ between them, then $\cos(\theta) = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\cdot\|\mathbf{w}\|}$ and so $\theta = \cos^{-1}\left(\dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\cdot\|\mathbf{w}\|}\right)$.

Example 1.7.7. Find the angle θ between the vectors $(1, 2, 3)$ and $(-2, 3, 0)$.

Solution

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{(1,2,3)\cdot(-2,3,0)}{\|(1,2,3)\|\cdot\|(-2,3,0)\|}\right)\\
&= \cos^{-1}\left(\frac{(1)(-2)+(2)(3)+(3)(0)}{\sqrt{1^2+2^2+3^2}\cdot\sqrt{(-2)^2+3^2+0^2}}\right)\\
&= \cos^{-1}\left(\frac{4}{\sqrt{14}\cdot\sqrt{13}}\right)\\
&\simeq 1.270 \text{ radians or } 72.75^\circ.
\end{aligned}$$

Also note that if the dot product between two non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ is zero then we must have $\cos(\theta) = 0$ (since $\|\mathbf{v}\|\cdot\|\mathbf{w}\| \neq 0$). This then means that θ must be $\pm\pi/2$, that is $\mathbf{v}$ and $\mathbf{w}$ are at right angles. Of course the opposite is also true, if $\mathbf{v}$ and $\mathbf{w}$ are not at right angles, then $\cos(\theta) \neq 0$, so that $\mathbf{v}\cdot\mathbf{w} \neq 0$. This gives us a very useful test for seeing if two vectors are at right angles to one another; we simply need to calculate the dot product of them and check if it is zero. Often we say that vectors that are at right angles to each other are *orthogonal* and we also use this word to describe vectors in higher dimensional spaces whose dot product is zero.

Remark 1.7.8. Useful properties of the dot product are

(1) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (it is commutative).
(2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ and $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ (it distributes over vector addition).
(3) If k is a real number then $\mathbf{v} \cdot (k\mathbf{w}) = (k\mathbf{v}) \cdot \mathbf{w} = k(\mathbf{v} \cdot \mathbf{w})$.

The other important way to combine vectors is the so called cross product. In contrast to the dot product, when we form the cross product between two vectors, we end up with another vector, rather than a number. Also in contrast to the dot product, we can only find the cross product of two vectors in $\mathbb{R}^3$ and it is defined as follows.

Definition 1.7.9 (Cross Product)**.** Let $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ be two vectors in $\mathbb{R}^3$. Then the *cross product* of $\mathbf{v}$ with $\mathbf{w}$ is defined to be

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the vectors of length one in the directions of the coordinate axes.

Example 1.7.10. Find the cross product of the vectors $(1, 2, 3)$ and $(-2, 3, 0)$.

Solution

$$\begin{aligned}(1,2,3) \times (-2,3,0) &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 3 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} \hat{k} \\ &= (2 \times 0 - 3 \times 3)\hat{\imath} - (1 \times 0 - 3 \times (-2))\hat{\jmath} + (1 \times 3 - 2 \times (-2))\hat{k} \\ &= -9\hat{\imath} - 6\hat{\jmath} + 7\hat{k}.\end{aligned}$$

The cross product also has a geometric meaning. Firstly we have that

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \cdot \|\mathbf{w}\| \sin(\theta),$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Thus we see that the cross product between two non-zero vectors is zero if and only if $\sin(\theta) = 0$, that is if and only if $\theta = 0$ or $\theta = \pi$, that is if and only if the vectors are parallel. However calculating the cross product is not a good way to check if two vectors are parallel, since this can easily be checked by seeing if one vector is a scalar multiple of the other.

The important geometric aspect of the of the cross product is that the resulting vector $\mathbf{v} \times \mathbf{w}$ is at right angles to both $\mathbf{v}$ and $\mathbf{w}$ (provided $\mathbf{v} \times \mathbf{w}$ is not the zero vector). This gives us a very useful way of constructing a vector at right angles to two given vectors. The actual direction of $\mathbf{v} \times \mathbf{w}$ can be determined using the 'screw rule'. This says that if you imagine turning a screw from $\mathbf{v}$ to $\mathbf{w}$ (through the smaller angle) then the direction the screw moves in is the direction of $\mathbf{v} \times \mathbf{w}$.

For example, to find the direction of $\mathbf{v} \times \mathbf{w}$ in Figure 2, to turn a screw from $\mathbf{v}$ to $\mathbf{w}$ we have to turn it anticlockwise and if we do that the screw will come out of the page. So that is the direction of $\mathbf{v} \times \mathbf{w}$.

Remark 1.7.11. A useful property of the cross product is $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$. This follows since if we swap two rows of a matrix, the sign of its determinant changes.

1.8. **Eigenvalues and Eigenvectors.**

If we have a $n \times 1$ column vector $\mathbf{v}$ and multiply it on the left by a $n \times n$ matrix A, then we will obtain another $n \times 1$ column vector $A\mathbf{v}$. In general this new vector will not be parallel to $\mathbf{v}$ but for certain vectors it may turn out that $\mathbf{v}$ and $A\mathbf{v}$ are parallel. That is, it may happen that $A\mathbf{v} = \lambda\mathbf{v}$ for some number λ. The vectors $\mathbf{v}$ where this happens and the corresponding λ's are very special and we have a name for them.

Definition 1.8.1 (Eigenvalues and Eigenvectors)**.** If A is a $n \times n$ matrix, if $\mathbf{v}$ is a $n \times 1$ non-zero column vector, if λ is a scalar and if $A\mathbf{v} = \lambda\mathbf{v}$ then we say that λ is an *eigenvalue* of A with $\mathbf{v}$ the corresponding *eigenvector*.

Since, if we multiply any vector by the identity matrix I (of the appropriate size), we get the same vector back, we can rewrite $A\mathbf{v} = \lambda\mathbf{v}$ as $A\mathbf{v} = \lambda I\mathbf{v}$. Collecting both vectors on the left side of the equation, we obtain $A\mathbf{v} - \lambda I\mathbf{v} = \mathbf{0}$. This can also be written as $(A - \lambda I)\mathbf{v} = \mathbf{0}$. Now if $A - \lambda I$ were invertible, we could multiply both sides of this equation on the left by $(A - \lambda I)^{-1}$ and obtain $\mathbf{v} = \mathbf{0}$. However in the definition, we have assumed that $\mathbf{v}$ is non-zero, so it must be the case that $A - \lambda I$ is not invertible. Thus we must have $\det(A - \lambda I) = 0$. The equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of the matrix A and it is this equation that we have to solve to find the eigenvalues λ. If A is a $n \times n$ matrix then $\det(A - \lambda I)$ will be a polynomial of degree n. In this course we will restrict ourselves to finding eigenvalues for 2×2 matrices, which will involve solving quadratic equations.

Example 1.8.2. Find the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$.

Solution We start by writing down the characteristic equation. In this case it is

$$\det\left(\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

We can write this as $\det\begin{pmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{pmatrix} = 0$ and on calculating the determinant we obtain $(1-\lambda)(-4-\lambda) - 6 = 0$ or $\lambda^2 + 3\lambda - 10 = 0$. Thus $(\lambda - 2)(\lambda + 5) = 0$, so that the eigenvalues are $\lambda = 2$ and $\lambda = -5$.

Warning 1.8.3. Since the characteristic equation for a 2×2 matrix is a quadratic equation, it may happen that we get repeated roots or we may get no real solutions. We will not examine the case where we get complex eigenvalues in this course but there will be examples where there are repeated eigenvalues. For example the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has a repeated eigenvalue $\lambda = 1$.

Once we have found the eigenvalues, we then have to find the eigenvectors corresponding to these eigenvalues. Note that if $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue λ and if α is any non-zero number, then $A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v})$. This tells us that if $\mathbf{v}$ is an eigenvector corresponding to λ, then $\alpha\mathbf{v}$ will also be an eigenvector corresponding to λ. In other words if $\mathbf{v}$ is an eigenvector corresponding to λ then any vector parallel to $\mathbf{v}$ will also be an eigenvector corresponding to λ. Now let us continue the previous example and find eigenvectors corresponding to the eigenvalues we have found.

Example 1.8.4. Find the eigenvectors corresponding to the eigenvalues of the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}$ found in Example 1.8.2.

Solution We will first find an eigenvector corresponding to the eigenvalue $\lambda = 2$. To do this we first form the *eigenvector equation* $A\mathbf{v} = \lambda\mathbf{v}$, where we let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$. Since $\lambda = 2$, this is $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$. On multiplying this out we obtain $\begin{pmatrix} x + 3y \\ 2x - 4y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$. Since two vectors are equal if and only if all their components are equal, from this one vector equation, we obtain the two ordinary equations $x + 3y = 2x$ and $2x - 4y = 2y$. Both these equations reduce to $x = 3y$, so that any non-zero vector of the form $\begin{pmatrix} 3a \\ a \end{pmatrix}$ will be an eigenvector corresponding to the eigenvalue $\lambda = 2$. If we just want one eigenvector, then we can let $a = 1$, say, to obtain the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Next we will find an eigenvector corresponding to the eigenvalue $\lambda = -5$. In this case the eigenvector equation becomes $\begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$. On multiplying this out we obtain $\begin{pmatrix} x + 3y \\ 2x - 4y \end{pmatrix} = \begin{pmatrix} -5x \\ -5y \end{pmatrix}$, which yields the two ordinary equations $x+3y = -5x$ and $2x-4y = -5y$. These both reduce to the equation $y = -2x$, so a suitable eigenvector corresponding to the eigenvalue $\lambda = -5$ is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Warning 1.8.5. It is a feature of the eigenvector equation that when you solve the pair of simultaneous equations you get from the two components of the vector equation you **MUST** end up with at least one free variable (this is since any non-zero multiple of an eigenvector is also an eigenvector). If this does not happen then you have gone wrong somewhere and you should have a look over your work to see if you can spot where. Putting it another way, if you end up by obtaining a specific vector $\begin{pmatrix} a \\ b \end{pmatrix}$ as the only solution to the eigenvector equation or you end up with no vector solutions, then this means you have gone wrong somewhere.

Also note that if we have a repeated eigenvalue, one of two things may happen. Firstly we might be able to find two eigenvectors in different directions and in this case **EVERY** non-zero vector is an eigenvector. For example this happens if we find the eigenvectors of the identity matrix. As we noted above this has only one eigenvalue $\lambda = 1$, so the eigenvector equation is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$ and on multiplying this out we just obtain $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ which gives no constraints at all on what x and y can be. In other words, every non-zero vector is an eigenvector. On the other hand it is possible that we may only be able to find one direction along which eigenvectors lie. For example if we find the eigenvalues of $\begin{pmatrix} -2 & 1 \\ -1 & -4 \end{pmatrix}$ we will see it has a repeated eigenvalue $\lambda = -3$. However if we then calculate the eigenvectors, we will see that they are all parallel to the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. When this happens we say that the matrix has a *defect* and there is a whole advanced area of study of matrices that are defective.