ACCESS TO SCIENCE, ENGINEERING AND AGRICULTURE: MATHEMATICS 2

MATH00040

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3. Differential Calculus

3.1. Introduction to Differentiation.

For most of this chapter we will concentrate on the mechanics of how to differentiate functions but before that we will quickly revise what it means to find the derivative of a function and what that derivative represents.

We will start with the formal definition.

Definition 3.1.1 (Derivative)**.** Let $f\colon (a, b) \to \mathbb{R}$, then the *derivative* of f at $x \in (a, b)$ is defined to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

if this limit exists.

Before we go any further let us examine what this definition means. If we look at Figure 1, we will see that the slope of the line from the point $(x, f(x))$ to the point $(x+h, f(x+h))$ is just $\dfrac{f(x+h) - f(x)}{h}$.

When we differentiate a function at a point x, what we are really doing is to make h smaller and smaller in $\dfrac{f(x+h) - f(x)}{h}$ and see what happens to it. What this means graphically is shown in Figure 2. Hopefully this figure will convince you that the derivative of f at x is the slope of the tangent line to the function f at the point x.

Remark 3.1.2. Often the derivative of a function f will be denoted by $\dfrac{dy}{dx}$, $\dfrac{df}{dx}$, $\dfrac{d}{dx}(f)$ or even f_x rather than $f'(x)$. Also note that sometimes completely different letters may be use, so you may see things like $g'(x)$ or $\dfrac{dg}{dx}$ or indeed $\dfrac{dx}{dy}$, where the roles of x and y have been reversed. All these different notations mean exactly the

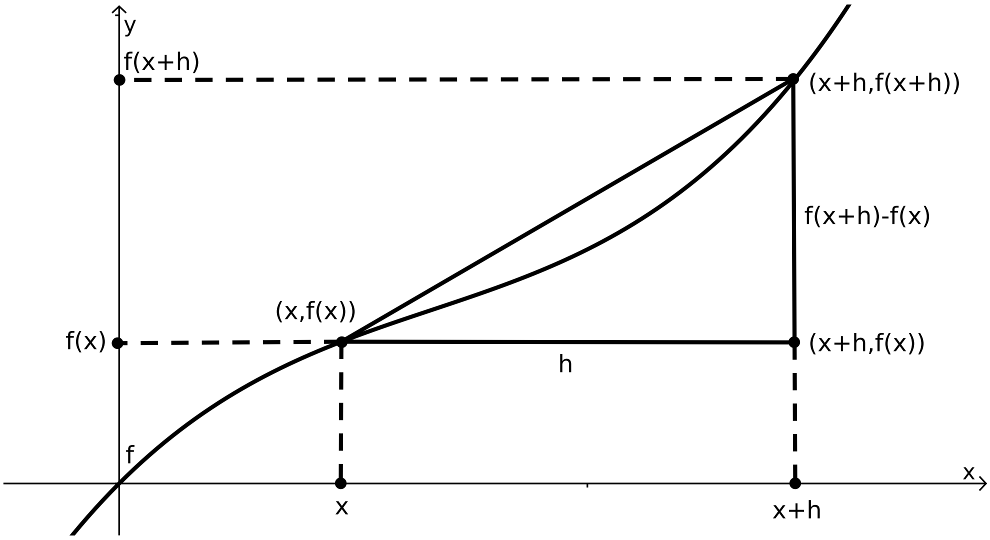

FIGURE 1. Slope of a line connecting $(x, f(x))$ to $(x + h, f(x + h))$.

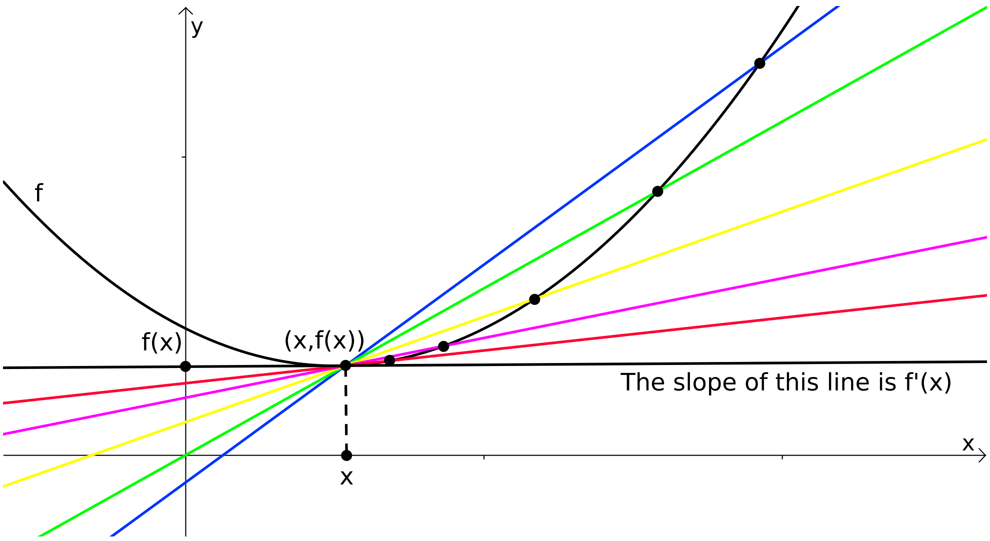

FIGURE 2. Geometric meaning of the derivative.

same thing. They only exist since calculus was developed by different mathematicians and the various notations have persisted.

While all this may seem like a lot of effort to go to just to find the gradient of a tangent to a curve, it is extremely important since it arises in numerous different areas. Whenever you want to find the rate of change of something then calculus will come in handy. For example, if you have a function representing the position of an object, then the derivative will represent the velocity of the object. Similarly if you have a function representing the velocity of an object then the derivative of this function will represent the acceleration of the object.

3.2. Some Common Derivatives.

As we noted in Section 3.1, we will concentrate on the actual mechanics of differentiation, rather than worrying about differentiating functions from first principles. In Table 1 there is a list of derivatives that you should be able to use. Note that a formula sheet will be provided in the exam, so you should concentrate on learning how to use them, not on memorising them.

$f(x)$	$f'(x)$	Comments
c	0	Here c is any real number
x^n	nx^{n-1}	
e^{ax}	ae^{ax}	
$\ln(ax)$	$\dfrac{1}{x}$	Here we must have $ax > 0$
$\sin(ax)$	$a\cos(ax)$	
$\cos(ax)$	$-a\sin(ax)$	Note the change of sign

TABLE 1. Some common derivatives

Warning 3.2.1.
(1) Note that the derivative of $\ln(ax)$ is $\dfrac{1}{x}$, no matter what the value of a is (provided $ax > 0$). This is **NOT** a typo.
(2) Also note that the derivatives of $\sin(ax)$ and $\cos(ax)$ are only valid if x is in radians. If x is in degrees then extra constants are needed.

As usual, a few examples will make things clearer. Please see Table 2.

3.3. The Sum and Multiple Rules.

Although the list of derivatives in Table 1 is very useful, we would not get very far if these were the only functions we could differentiate. Luckily there are rules that allow us to differentiate more complicated functions. The first of these allows us to differentiate sums of functions.

Theorem 3.3.1 (The Sum Rule for Differentiation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the derivative of $f+g$ at $x \in (a,b)$ is given by*

$$(f+g)'(x) = f'(x) + g'(x),$$

provided these derivatives exist.

All this says is that if we want to differentiate a sum of two functions then all we have to do is differentiate them separately and add the derivatives.

$f(x)$	$f'(x)$	Comments
0	0	Note the derivative of 0 is 0
2	0	
-4	0	
$-\pi$	0	$-\pi$ is just a number
e	0	e is just a number
$\cos(1)$	0	$\cos(1)$ is just a number
x	1	Since $x = x^1$, $n = 1$ giving $1x^0 = 1$
x^3	$3x^2$	Here we take $n = 3$
x^{-4}	$-4x^{-5} = -\dfrac{4}{x^5}$	Here we take $n = -4$
x^π	$\pi x^{\pi-1}$	π is just a number
x^{-e}	$-ex^{-e-1} = -\dfrac{e}{x^{e+1}}$	e is just a number
e^x	e^x	Here we take $a = 1$
e^{5x}	$5e^{5x}$	Here we take $a = 5$
e^{-7x}	$-7e^{-7x}$	Here we take $a = -7$
e^{ex}	$e \cdot e^{ex} = e^{ex+1}$	Here we take $a = e$
$\ln(x)$	$\dfrac{1}{x}$	Here we must have $x > 0$
$\ln(5x)$	$\dfrac{1}{x}$	Here we must have $x > 0$
$\ln(-5x)$	$\dfrac{1}{x}$	Here we must have $x < 0$
$\sin(x)$	$\cos(x)$	Here we take $a = 1$
$\sin(3x)$	$3\cos(3x)$	Here we take $a = 3$
$\sin(-2x)$	$-2\cos(-2x)$	Here we take $a = -2$
$\sin(-\pi x)$	$-\pi\cos(-\pi x)$	Here we take $a = -\pi$
$\cos(x)$	$-\sin(x)$	Here we take $a = 1$
$\cos(4x)$	$-4\sin(4x)$	Here we take $a = 4$
$\cos(-5x)$	$5\sin(-5x)$	Note $-(-5) = +5$
$\cos(\pi x)$	$-\pi\sin(\pi x)$	Here we take $a = \pi$

TABLE 2. Some examples of derivatives

Here are a couple of examples of the use of the Sum Rule.

Example 3.3.2. (1) Find the derivative of $f(x) = x^2 + \sin(2x)$.

$$f'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin(2x)) = 2x + 2\cos(2x).$$

(2) Find the derivative of $f(x) = \ln(2x) + e^{-3x}$.
Provided $x > 0$ (so that the derivative of the first term exists),

$$f'(x) = \frac{d}{dx}(\ln(2x)) + \frac{d}{dx}(e^{-3x}) = \frac{1}{x} - 3e^{-3x}.$$

The second rule that will enable us to differentiate a larger range of functions is the Multiple Rule.

Theorem 3.3.3 (The Multiple Rule for Differentiation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $c \in \mathbb{R}$, then the derivative of cf at $x \in (a,b)$ is given by*

$$(cf)'(x) = cf'(x),$$

provided the derivative of f exists.

All this says is that if we want to differentiate a constant multiple of a function, then all we have to do is first differentiate the function and then multiply by the constant.

Here are a couple of examples of the Multiple Rule.

Example 3.3.4. (1) Find the derivative of $f(x) = 5x^3$.

$$f'(x) = 5 \times \frac{d}{dx}(x^3) = 5 \times 3x^2 = 15x^2.$$

(2) Find the derivative of $f(x) = -3\cos(2x)$.

$$f'(x) = -3 \times \frac{d}{dx}(\cos(2x)) = -3 \times (-2\sin(2x)) = 6\sin(2x).$$

Warning 3.3.5. The Multiple Rule can only be used to differentiate a product of a number and a function. If we want to differentiate the product of two functions, then we have to use the Product Rule which is introduced in Section 3.4

Of course we are free to use both the Sum and Multiple Rules to differentiate a function and the following are a couple of examples of this.

Example 3.3.6. (1) Find the derivative of $f(x) = 5x^2 - 4x + 3$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(5x^2) + \frac{d}{dx}(-4x) + \frac{d}{dx}(3) \quad \text{(using the Sum Rule)}\\
&= 5\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(3) \quad \text{(using the Multiple Rule)}\\
&= 5(2x) - 4(1) + 0\\
&= 10x - 4.
\end{aligned}$$

(2) Find the derivative of $f(x) = -e^{-2x} - 2\cos(-3x)$.

$$\begin{aligned}
f'(x) &= \frac{d}{dx}(-e^{-2x}) + \frac{d}{dx}(-2\cos(-3x)) \quad \text{(using the Sum Rule)}\\
&= -\frac{d}{dx}(e^{-2x}) + (-2)\frac{d}{dx}(\cos(-3x)) \quad \text{(using the Multiple Rule)}\\
&= -(-2e^{-2x}) - 2(-3(-\sin(-3x)))\\
&= 2e^{-2x} - 6\sin(-3x).
\end{aligned}$$

3.4. The Product Rule.

The next rule that we will introduce is the Product Rule, which will enable us to differentiate a function which may be regarded as a product of two simpler functions.

Theorem 3.4.1 (The Product Rule for Differentiation)**.** *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the derivative of fg at $x \in (a,b)$ is given by*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

provided these derivatives exist.

That is, to differentiate a product, we add the derivative of the first times the second to the derivative of the second times the first.

As always, a couple of examples will say more than words.

Example 3.4.2. (1) Find the derivative of $f(x) = x^2 \sin(x)$.

$$f'(x) = \frac{d}{dx}(x^2)\sin(x) + x^2\frac{d}{dx}(\sin(x)) = 2x\sin(x) + x^2\cos(x).$$

(2) Find the derivative of $f(x) = e^{-3x}\ln(2x)$.
Provided $x > 0$ (so that the derivative of the $\ln(2x)$ exists),

$$f'(x) = \frac{d}{dx}(e^{-3x})\ln(2x) + e^{-3x}\frac{d}{dx}(\ln(2x)) = -3e^{-3x}\ln(2x) + e^{-3x}\frac{1}{x}.$$

Of course we can combine any of the Sum, Multiple or Product Rules to differentiate a function. The following are a couple of examples of this.

Example 3.4.3. (1) Find the derivative of $f(x) = -3x^4\sin(4x) + 5\sin(x)e^{2x}$.

$$\begin{aligned}
f'(x) =& \frac{d}{dx}(-3x^4\sin(4x)) + \frac{d}{dx}(5\sin(x)e^{2x}) \quad \text{(using the Sum Rule)}\\
=& -3\frac{d}{dx}(x^4\sin(4x)) + 5\frac{d}{dx}(\sin(x)e^{2x}) \quad \text{(using the Multiple Rule)}\\
=& -3\left(\frac{d}{dx}(x^4)\sin(4x) + x^4\frac{d}{dx}(\sin(4x))\right)\\
&+5\left(\frac{d}{dx}(\sin(x))e^{2x} + \sin(x)\frac{d}{dx}(e^{2x})\right) \quad \text{(using the Product Rule)}\\
=& -3\left(4x^3\sin(4x) + x^4(4\cos(4x))\right) + 5\left(\cos(x)e^{2x} + \sin(x)(2e^{2x})\right)\\
=& -12x^3\sin(4x) - 12x^4\cos(4x) + 5\cos(x)e^{2x} + 10\sin(x)e^{2x}.
\end{aligned}$$

(2) Find the derivative of $f(x) = 2\sin(2x)\cos(-x) - 3e^{-2x}\ln(4x)$.
Provided $x > 0$ (so that the derivative of the $\ln(4x)$ exists),

$$\begin{aligned}
f'(x) =& \frac{d}{dx}(2\sin(2x)\cos(-x)) + \frac{d}{dx}(-3e^{-2x}\ln(4x)) \quad \text{(using the Sum Rule)}\\
=& 2\frac{d}{dx}(\sin(2x)\cos(-x)) - 3\frac{d}{dx}(e^{-2x}\ln(4x)) \quad \text{(using the Multiple Rule)}\\
=& 2\left(\frac{d}{dx}(\sin(2x))\cos(-x) + \sin(2x)\frac{d}{dx}(\cos(-x))\right)\\
&- 3\left(\frac{d}{dx}(e^{-2x})\ln(4x) + e^{-2x}\frac{d}{dx}(\ln(4x))\right) \quad \text{(using the Product Rule)}\\
=& 2\left(2\cos(2x)\cos(-x) + \sin(2x)(-(-\sin(-x)))\right)\\
&- 3\left(-2e^{-2x}\ln(4x) + e^{-2x}\frac{1}{x}\right)\\
=& 4\cos(2x)\cos(-x) + 2\sin(2x)\sin(-x) + 6e^{-2x}\ln(4x) - 3e^{-2x}\frac{1}{x}.
\end{aligned}$$

3.5. **The Quotient Rule**.

The next rule that we will need is the Quotient Rule, which will enable us to differentiate a function which may be regarded as a quotient of two simpler functions.

Theorem 3.5.1 (The Quotient Rule for Differentiation). *Let $f\colon (a,b) \to \mathbb{R}$ and $g\colon (a,b) \to \mathbb{R}$, then the derivative of $\frac{f}{g}$ at $x \in (a,b)$ is given by*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)},$$

provided these derivatives exist and provided that $g(x) \neq 0$.

I don't think it adds anything to express this theorem in words but a couple of examples will be helpful.

Example 3.5.2. (1) Find the derivative of $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

Provided $\cos(x) \neq 0$,

$$\begin{aligned} f'(x) =& \frac{\frac{d}{dx}(\sin(x))\cos(x) - \sin(x)\frac{d}{dx}(\cos(x))}{\cos^2(x)} \\ =& \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ =& \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ =& \frac{1}{\cos^2(x)} \\ =& \sec^2(x). \end{aligned}$$

(2) Find the derivative of $f(x) = \dfrac{e^x}{x^2}$.

$$\begin{aligned} f'(x) =& \frac{\frac{d}{dx}(e^x)x^2 - e^x\frac{d}{dx}(x^2)}{(x^2)^2} \\ =& \frac{e^x x^2 - e^x(2x)}{x^4} \\ =& \frac{e^x x^2 - 2xe^x}{x^4} \end{aligned}$$

Of course, the Quotient Rule can also be combined with the Sum, Multiple or Product Rules as necessary.

3.6. **The Chain Rule.**

The last rule that we will need is the Chain Rule (also called the Composition Rule). This enables us to differentiate functions of the form $f(x) = \sin(x^3)$ where there is an 'outer' function (in this case $\sin(x)$) which is a function of an 'inner' function (in this case x^3).

Theorem 3.6.1 (The Chain Rule for Differentiation)**.** *Let u be a function of x and y be a function of u, then the derivative of $y = f(x)$ is given by*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

provided these derivatives exist.

Remark 3.6.2. This theorem can also be written in notation similar to the previous theorems in this section (as $(f \circ g)'(x) = (f' \circ g)(x) \cdot g'(x)$). However I don't recommend this latter equation when actually differentiating, it is much easier to use the equation in the theorem.

A couple of examples will illustrate the method.

Example 3.6.3. (1) Find the derivative of $f(x) = \sin(x^3)$.

In this case we let $u = x^3$ and $y = \sin(u)$. Then $\dfrac{du}{dx} = 3x^2$ and $\dfrac{dy}{du} = \cos(u)$. Hence

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u)(3x^2) = 3x^2 \cos(x^3).$$

(2) Find the derivative of $f(x) = e^{\cos(x)}$.

In this case we let $u = \cos(x)$ and $y = e^u$. Then $\dfrac{du}{dx} = -\sin(x)$ and $\dfrac{dy}{du} = e^u$. Hence

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u(-\sin(x)) = -\sin(x)e^{\cos(x)}.$$

Warning 3.6.4. When you have obtained the derivative in terms of x and u, you still have to put u in terms of x. Remember that u was not in the original function, we just introduced it to enable us to perform the differentiation.

Of course, the Chain Rule can be combined with the Sum, Multiple, Product or Quotient Rules as necessary and indeed to differentiate a particular function, you may have to use some of these rules more than once.

3.7. **Critical Points.**

We noted in Section 3.1 that the derivative of a function tells us the gradient of the tangent to the graph at a particular point. The points where this tangent is horizontal have a special name.

Definition 3.7.1 (Critical Point)**.** Let f be a function, then a is said to be a *critical point* of f if $f'(a) = 0$.

Critical points often have a special physical significance and this is one reason why we often want to find them. For example, if f represents the position of an object as a function of time, then the critical points of f are the times when the velocity is zero (i.e., object is not moving). Similarly, if f represents the velocity of an object as a function of time, then the critical points of f are the times when the object is not accelerating (i.e., the velocity is not changing).

To find the critical points we have to solve the equation $f'(x) = 0$ and the following are a couple of examples.

Example 3.7.2. (1) Find the critical points of $f(x) = 2x^3 - 9x^2 + 12x + 5$.

Since $f'(x) = 6x^2 - 18x + 12$, to find the critical points, we have to solve the equation $6x^2 - 18x + 12 = 0$. But

$$\begin{aligned} 6x^2 - 18x + 12 = 0 &\Leftrightarrow x^2 - 3x + 2 = 0 \\ &\Leftrightarrow (x-1)(x-2) = 0 \\ &\Leftrightarrow x = 1 \text{ or } x = 2. \end{aligned}$$

Thus the critical points of f are $x = 1$ and $x = 2$. These are shown in Figure 3.

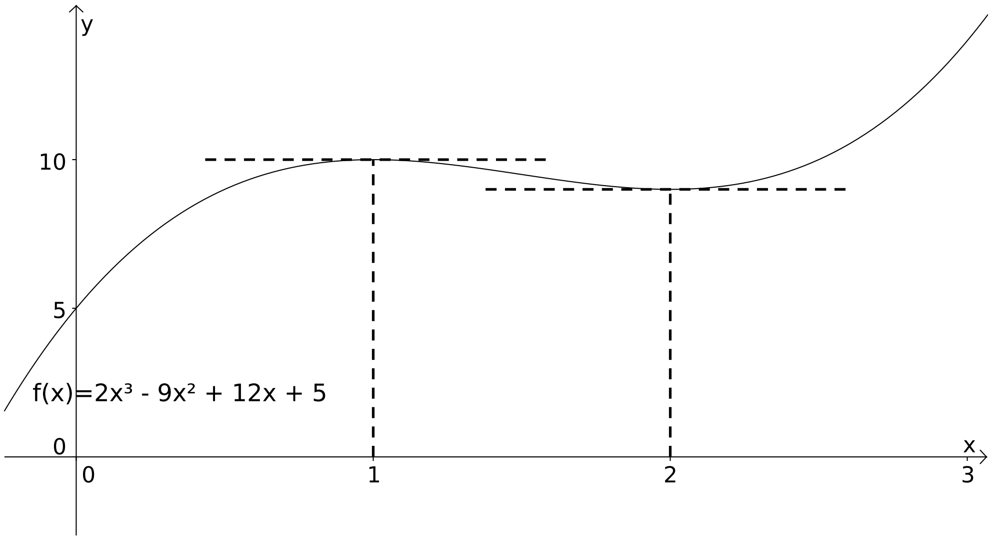

FIGURE 3. Critical points of the function $f(x) = 2x^3 - 9x^2 + 12x + 5$.

(2) Find the critical points of $f(x) = \cos(x)$.
Here $f'(x) = -\sin(x)$, so the critical points of f are the points where $-\sin(x) = 0$, that is where $\sin(x) = 0$. Thus the critical points are $x = k\pi$, where k is any integer. Some of these are shown in Figure 4.

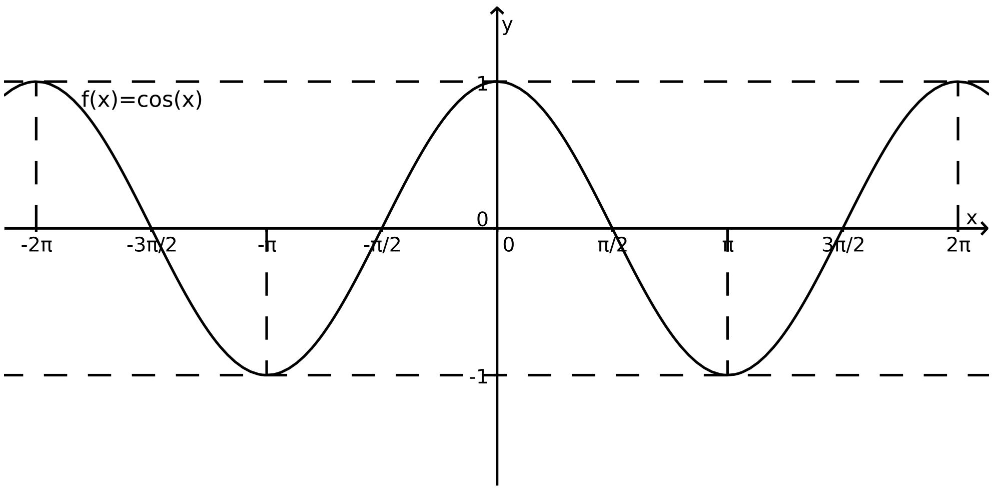

FIGURE 4. Some of the critical points of the function $f(x) = \cos(x)$.

Remark 3.7.3. In Example 3.7.2.1, f had two critical points while in Example 3.7.2.2, f had an infinite number of critical points. In general a function can have any number of critical points (including 0).

Example 3.7.4. Find the critical points of the function $f(x) = e^{2x} - 5x$.
Since $f'(x) = 2e^{2x} - 5$, to find the critical points, we have to solve the equation $2e^{2x} - 5 = 0$. But

$$\begin{aligned} 2e^{2x} - 5 = 0 &\Leftrightarrow 2e^{2x} = 5 \\ &\Leftrightarrow e^{2x} = \frac{5}{2} \\ &\Leftrightarrow \ln\left(e^{2x}\right) = \ln\left(\frac{5}{2}\right) \\ &\Leftrightarrow 2x = \ln\left(\frac{5}{2}\right) \\ &\Leftrightarrow x = \frac{1}{2}\ln\left(\frac{5}{2}\right). \end{aligned}$$

So there is one critical point of f, that is $x = \frac{1}{2}\ln\left(\frac{5}{2}\right)$. This shown in Figure 5.

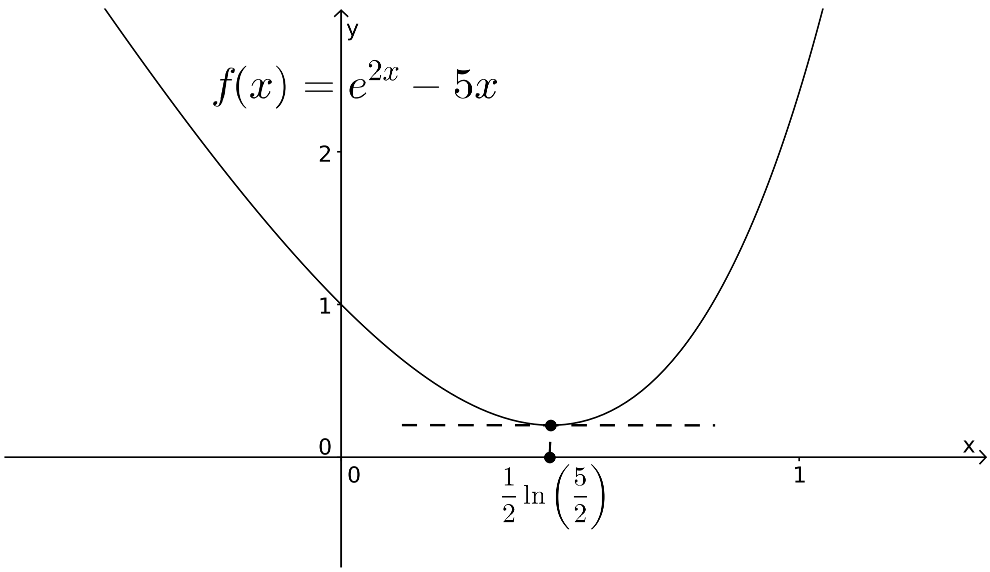

FIGURE 5. Critical point of the function $e^{2x} - 5x$.

Warning 3.7.5. Note the question asks for the 'critical points' even though it turns out in the end that there is only one. So you should not read anything in to whether or not there is a plural in the question. There could turn out to be any number of critical points in such a case (including zero).

Here are a couple of examples where a function has no critical points.

Example 3.7.6. Find the critical points of the function $f(x) = e^{-3x} - 4x$.
Since $f'(x) = -3e^{-3x} - 4$, to find the critical points, we have to solve the equation $-3e^{-3x} - 4 = 0$.
But

$$\begin{aligned} -3e^{-3x} - 4 = 0 &\Leftrightarrow -3e^{-3x} = 4 \\ &\Leftrightarrow e^{-3x} = -\frac{4}{3}. \end{aligned}$$

However $e^{-3x} > 0$, so that $e^{-3x} = -\dfrac{4}{3}$ and hence $-3e^{-3x} - 4 = 0$ have no solutions. Thus f has no critical points.

Remark 3.7.7. Since $e^{-3x} > 0$, we have $-3e^{-3x} < 0$ and hence $-3e^{-3x} - 4 < 0$. This means that the gradient of the function $f(x) = e^{-3x} - 4x$ is always negative and this can be seen in Figure 6.

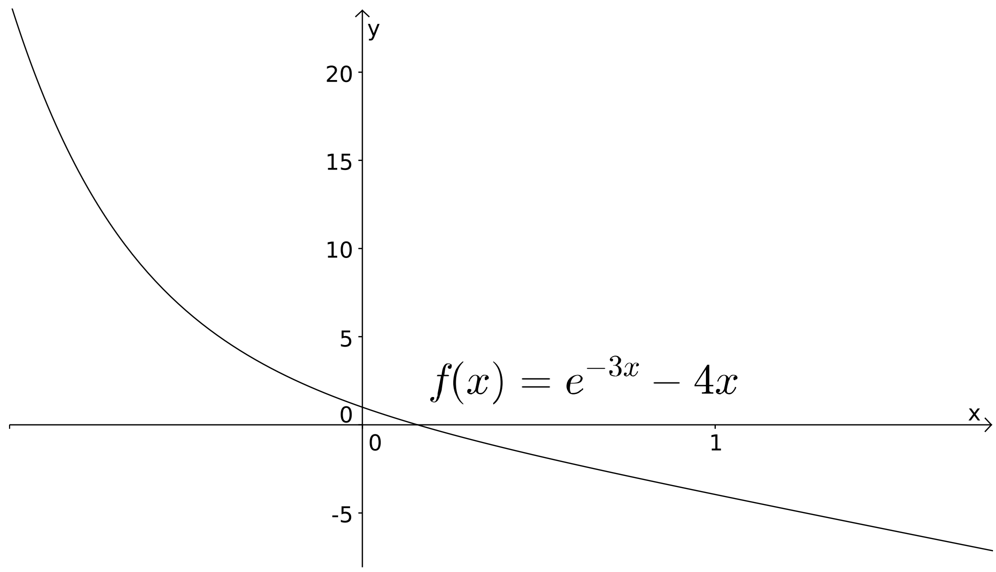

FIGURE 6. Graph of the function $f(x) = e^{-3x} - 4x$.

Example 3.7.8. Find the critical points of the function $f(x) = 2x^3 + 3x^2 + 6x - 5$.
Since $f'(x) = 6x^2 + 6x + 6$, to find the critical points, we have to solve the equation $6x^2 + 6x + 6 = 0$. But

$$6x^2 + 6x + 6 = 0 \Leftrightarrow x^2 + x + 1 = 0.$$

Now the discriminant of the expression $x^2 + x + 1$ is

$$b^2 - 4ac = 1^2 - 4(1)(1) = 1 - 4 = -3,$$

and since this is negative, the equation $x^2 + x + 1 = 0$ has no (real) solutions.
Hence the function $f(x) = 2x^3 + 3x^2 + 6x - 5$ has no critical points.

Remark 3.7.9. In fact the gradient of the function $f(x) = 2x^3 + 3x^2 + 6x - 5$ is always positive and this can be seen in Figure 7.

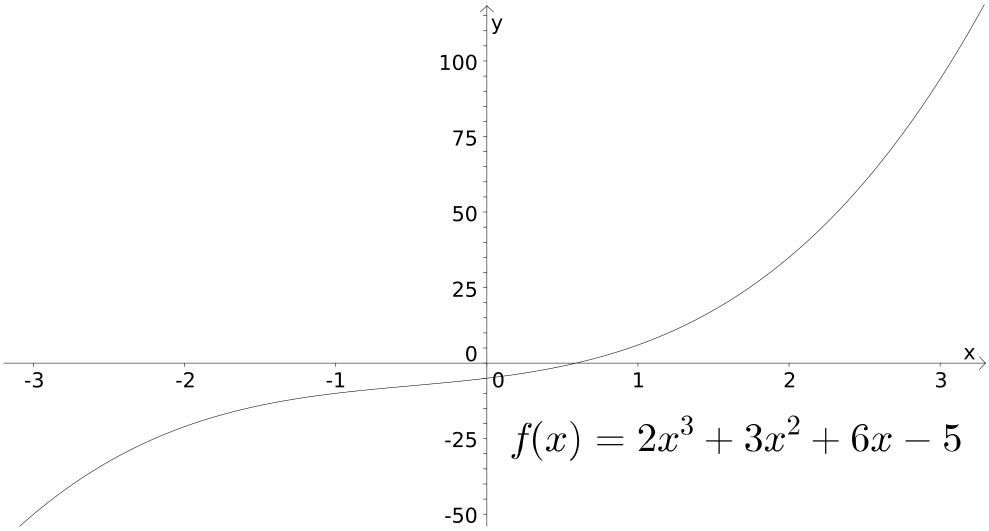

FIGURE 7. Graph of the function $f(x) = 2x^3 + 3x^2 + 6x - 5$.

3.8. **Finding Maxima and Minima**.

Critical points are important since functions often attain their maximum or minimum at them. However they are not the only places where functions attain maxima and minima and so, before we go on to describe how to classify critical points, we will describe two different sorts of maximum and minimum and give another method of finding them. We will first define exactly what we mean by maxima and minima.

Definition 3.8.1 (Global maximum)**.** Given a set S and a function $f\colon S \to \mathbb{R}$, then we say f attains a *global maximum* at $a \in S$ if $f(x) \leqslant f(a)$ for all $x \in S$.

Definition 3.8.2 (Global minimum)**.** Given a set S and a function $f\colon S \to \mathbb{R}$, then we say f attains a *global minimum* at $a \in S$ if $f(a) \leqslant f(x)$ for all $x \in S$.

Definition 3.8.3 (Local maximum)**.** Given a set S and a function $f\colon S \to \mathbb{R}$, then we say f attains a *local maximum* at $a \in S$ if there exists some number b such that $f(x) \leqslant f(a)$ for all $x \in (a-b, a+b) \cap S$.

Definition 3.8.4 (Local minimum)**.** Given a set S and a function $f\colon S \to \mathbb{R}$, then we say f attains a *local minimum* at $a \in S$ if there exists some number b such that $f(a) \leqslant f(x)$ for all $x \in (a-b, a+b) \cap S$.

Remark 3.8.5. Note the use of the $\leqslant$ symbol in the above definitions. If we replace $\leqslant$ with $<$, then we say that the maxima or minima are *strict*.

As usual, a diagram will make it more obvious what all these definitions mean. All the global and local maxima and minima of the function

$$f\colon [0,3] \to \mathbb{R}$$
$$x \mapsto 2x^3 - 9x^2 + 12x + 5$$

are shown in Figure 8.

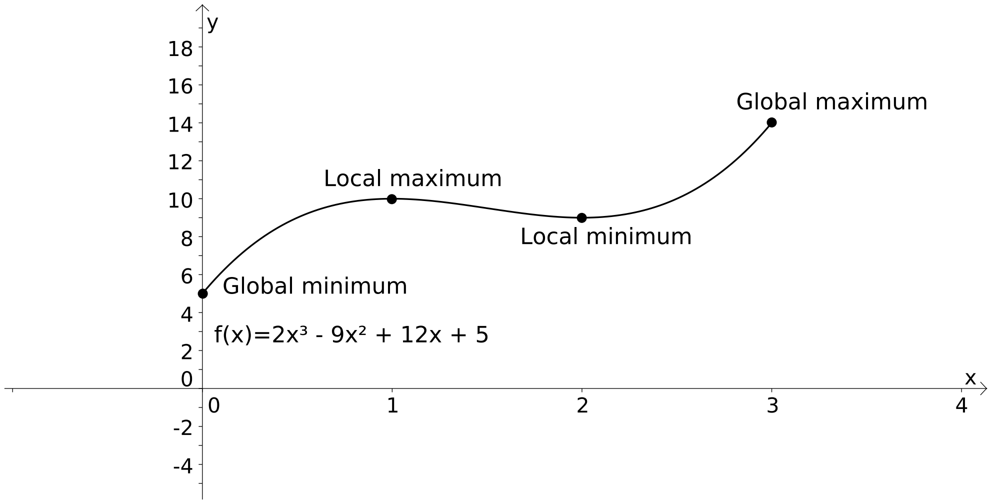

FIGURE 8. Global and local maxima and minima of the function $f(x) = 2x^3 - 9x^2 + 12x + 5$ with domain $[0, 3]$.

Note that in this particular case the global maxima and minima do not occur at the critical points, rather they occur at the end points of the domain. However if we just change the domain of the function to $[0.9, 2.1]$, say, then we have the situation shown in Figure 9.

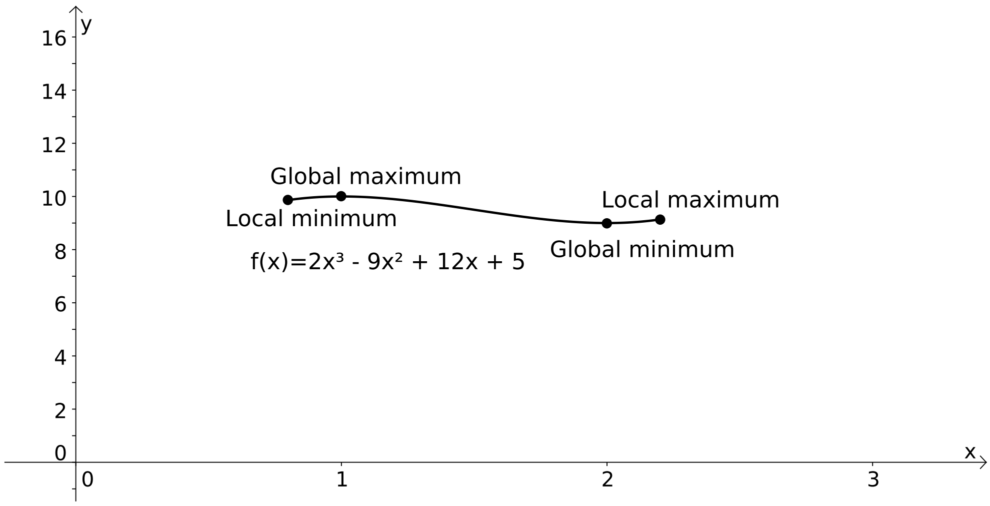

FIGURE 9. Global and local maxima and minima of the function $f(x) = 2x^3 - 9x^2 + 12x + 5$ with domain $[0.8, 2.2]$.

On the other hand, if we change the domain to $[0.5, 2.5]$, then it turns out that we have two global maximum points and two global minimum points. This is shown in Figure 10.

If we just want to find the global maxima and minima of a differentiable function that is defined on an interval $[a, b]$, then we just have to evaluate the function at all the critical points and at the endpoints of the domain. To see how this works in practice

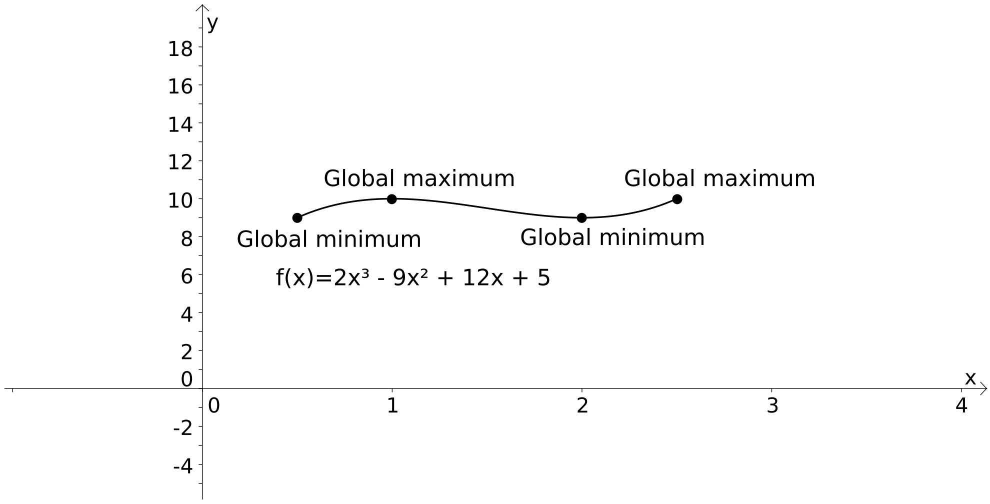

FIGURE 10. Global and local maxima and minima of the function $f(x) = 2x^3 - 9x^2 + 12x + 5$ with domain $[0.5, 2.5]$.

let us look at four examples involving the function with rule $f(x) = 2x^3 - 9x^2 + 12x + 5$ but with four different domains.

Example 3.8.6. (1) Find the global maxima and minima of the function

$$\begin{aligned} f\colon [0,3] &\to \mathbb{R} \\ x &\mapsto 2x^3 - 9x^2 + 12x + 5. \end{aligned}$$

We already know from Example 3.7.2.1 that the critical points of f are $x = 1$ and $x = 2$ (and these are in the domain). The endpoints of the domain are $x = 0$ and $x = 3$, so to find the global maxima and minima, we evaluate f at the above four points. $f(0) = 5$, $f(1) = 10$, $f(2) = 9$ and $f(3) = 14$. The smallest of these numbers is 5, so the global minimum of f occurs at $x = 0$. The largest of these numbers is 14, so the global maximum of f occurs at $x = 3$.

(2) Find the global maxima and minima of the function

$$\begin{aligned} f\colon [0.8, 2.2] &\to \mathbb{R} \\ x &\mapsto 2x^3 - 9x^2 + 12x + 5. \end{aligned}$$

The only difference here is that the endpoints of the domain are $x = 0.8$ and $x = 2.2$, so we use these two points instead of $x = 0$ and $x = 3$. $f(0.8) = 9.864$, $f(1) = 10$, $f(2) = 9$ and $f(2.2) = 9.136$. The smallest of these numbers is 9, so the global minimum of f occurs at $x = 2$. The largest of these numbers is 10, so the global maximum of f occurs at $x = 1$.

(3) Find the global maxima and minima of the function

$$f\colon [1.5, 3] \to \mathbb{R}$$
$$x \mapsto 2x^3 - 9x^2 + 12x + 5.$$

In this case we ignore the critical point at $x = 1$ since f is not defined there (really the critical point doesn't exist). So we evaluate f at the two endpoints $x = 1.5$ and $x = 3$, and the critical point $x = 2$.
$f(1.5) = 9.5$, $f(2) = 9$ and $f(3) = 14$.
The smallest of these numbers is 9, so the global minimum of f occurs at $x = 2$ and is $f(2) = 9$.
The largest of these numbers is 14, so the global maximum of f occurs at $x = 3$ and is $f(3) = 14$.

(4) Find the global maxima and minima of the function

$$f\colon [2.5, 3] \to \mathbb{R}$$
$$x \mapsto 2x^3 - 9x^2 + 12x + 5.$$

In this case we ignore both the critical points since f is not defined at either of them (again really the critical points don't exist). So we evaluate f at the two endpoints $x = 2.5$ and $x = 3$.
$f(2.5) = 10$ and $f(3) = 14$. The smallest of these numbers is 10, so the global minimum of f occurs at $x = 2.5$ and is $f(2.5) = 10$. The largest of these numbers is 14, so the global maximum of f occurs at $x = 3$ and is $f(3) = 14$.

Remark 3.8.7. Example 3.8.6 shows that the global maxima and minima depend not only on the rule of the function but also on the range of x it is defined on.

Warning 3.8.8.

- The requirement that the function is differentiable is essential for this technique to work. You will not need to worry about this in MATH00040 since I will always give you functions that are differentiable, but it might be important to bear this in mind in future courses.
- Remember that we only calculate the value of the function at a critical point if it lies in the interval where the function is defined. This is something that often causes errors.

Here are some more examples that involve a function that is not a polynomial.

Example 3.8.9.

(a) Find the global maxima and minima of the function $f(x) = e^{2x} - 5x$ defined for all x with $0 \leqslant x \leqslant 1$.
We already know from Example 3.7.4 that the critical point of f is $x = \frac{1}{2}\ln\left(\frac{5}{2}\right) \simeq 0.46$ and this satisfies $0 \leqslant x \leqslant 1$. The endpoints of the interval where the function is defined are $x = 0$ and $x = 1$, so to find the global maxima and minima, we evaluate f at the above three points.
$f(0) = e^0 - 0 = 1$,

$$f\left(\frac{1}{2}\ln\left(\frac{5}{2}\right)\right) = \exp\left(\ln\left(\frac{5}{2}\right)\right) - \frac{5}{2}\ln\left(\frac{5}{2}\right) = \frac{5}{2} - \frac{5}{2}\ln\left(\frac{5}{2}\right) \simeq 0.21,$$

and $f(1) = e^2 - 5 \simeq 2.39$.
The smallest of these values is $\frac{5}{2} - \frac{5}{2}\ln\left(\frac{5}{2}\right)$, so the global minimum of f occurs at $x = \frac{1}{2}\ln\left(\frac{5}{2}\right)$ and is $f\left(\frac{1}{2}\ln\left(\frac{5}{2}\right)\right) = \frac{5}{2} - \frac{5}{2}\ln\left(\frac{5}{2}\right)$.
The largest of these numbers is $e^2 - 5$, so the global maximum of f occurs at $x = 1$ and is $f(1) = e^2 - 5$.

Remark 3.8.10. I had to use a calculator to see which of the numbers was the smallest and which was the largest, but I still gave the answers using the exact values.

(b) Find the global maxima and minima of the function $f(x) = e^{2x} - 5x$ defined for all x with $1 \leqslant x \leqslant 2$.
In this case there is no critical point since the function is not defined when $x = \frac{1}{2}\ln\left(\frac{5}{2}\right) \simeq 0.46$ The endpoints of the interval where the function is defined are $x = 1$ and $x = 2$, so to find the global maxima and minima, we evaluate f at $x = 1$ and $x = 2$.
$f(1) = e^2 - 5 \simeq 2.39$ and $f(2) = e^4 - 10 \simeq 44.60$.
The smallest of these values is $e^2 - 5$, so the global minimum of f occurs at $x = 1$ and is $f(1) = e^2 - 5$.
The largest of these numbers is $e^4 - 10$, so the global maximum of f occurs at $x = 2$ and is $f(2) = e^4 - 10$.

3.9. **Classifying Critical Points.**

It may be that we are only interested in looking at the behaviour of a function near a critical point and this section gives one method of determining this in many cases. Since the derivative of a function is also a function, we can differentiate it again.

Definition 3.9.1 (Second Derivative)**.** The *second derivative* of a function f is defined to be the derivative of $f'(x)$.

Remark 3.9.2. The second derivative of a function f may be denoted by $f''(x)$ or $f^{(2)}(x)$ or $\dfrac{d^2f}{dx^2}$ or $\dfrac{d^2y}{dx^2}$ or f_{xx}

Theorem 3.9.3 (The Second Derivative Test)**.** *Let S be a set and let $f\colon S \to \mathbb{R}$ have a critical point at $a \in S$. Then there is a local maximum at a if $f''(a) < 0$ and there is a local minimum at a if $f''(a) > 0$.*

Warning 3.9.4. Note that the test does not mention the case when $f''(a) = 0$. If it does happen that $f''(a) = 0$ then the test does not tell us anything at all. In this case we can **NOT** conclude that f does not have a local maximum or local minimum at the point, it just means that the test doesn't work! For example, if $f(x) = x^4$, then $f''(0) = 0$ but f has a local minimum at $x = 0$. Similarly if $f(x) = -x^4$, then $f''(0) = 0$ but f has a local maximum at $x = 0$.

Remark 3.9.5. In fact the points where $f''(x) = 0$, with the sign of $f''(x)$ changing as we pass through the point, have a special name, they are called *points of inflection.* They may or not be critical points. For example $f(x) = \sin(x)$ has a point of inflection at $x = 0$ but this is not a critical point ($f'(0) = \cos(0) = 1 \neq 0$ while $f''(0) = -\sin(0) = 0$).

Let us now give an example to see how the Second Derivative Test works with our function $f(x) = 2x^3 - 9x^2 + 12x + 5$ (where this time we can take the domain to be $\mathbb{R}$).

Example 3.9.6. Classify the critical points of $f(x) = 2x^3 - 9x^2 + 12x + 5$.
We already know from Example 3.7.2.1 that $f'(x) = 6x^2 - 18x + 12$ and that f has critical points at $x = 1$ and $x = 2$. Now to find the second derivative we differentiate f'. We have $f''(x) = 12x - 18$, so that $f''(1) = -6$ and $f''(2) = 6$. Since $f''(1) < 0$, f has a local maximum at $x = 1$ and since $f''(2) > 0$, f has a local minimum at $x = 2$.

Our final example will deal with the function $f(x) = e^{2x} - 5x$ that we looked at in Example 3.7.4 and Example 3.8.9 (where we now assume f is defined everywhere).

Example 3.9.7. Classify the critical points of $f(x) = e^{2x} - 5x$.
We already know from Example 3.7.4 that $f'(x) = 2e^{2x} - 5$ and that f has one critical point at $x = \dfrac{1}{2}\ln\left(\dfrac{5}{2}\right)$.
Now to find the second derivative we differentiate f'.
We have $f''(x) = 4e^{2x}$, so that $f''\left(\dfrac{1}{2}\ln\left(\dfrac{5}{2}\right)\right) = 4\exp\left(\ln\left(\dfrac{5}{2}\right)\right) = 4\left(\dfrac{5}{2}\right) = 10$.
Since $f''\left(\dfrac{1}{2}\ln\left(\dfrac{5}{2}\right)\right) > 0$, f has a local minimum at $x = \dfrac{1}{2}\ln\left(\dfrac{5}{2}\right)$.

3.10. **The Newton-Raphson Method.**

In the final section of this chapter, we will use calculus to find approximate solutions to equations. In fact it is numerical methods like this that computer algebra packages (such as Maple and Mathematica) and indeed some programmable calculators use to solve equations.

The method enables us to find an approximate solution to the equation $f(x) = 0$ (and note we can always put an equation into this form by bringing all the terms over to one side of the equation). If we have an sufficiently good estimate (I won't go into what this means in this course) of the solution, say, x_n, then a better estimate of the solution is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

So what we do is to make a first guess at the solution, say x_0, and then successively get better ones x_1, x_2, etc. Figure 11 shows the sort of thing that happens (if we are lucky).

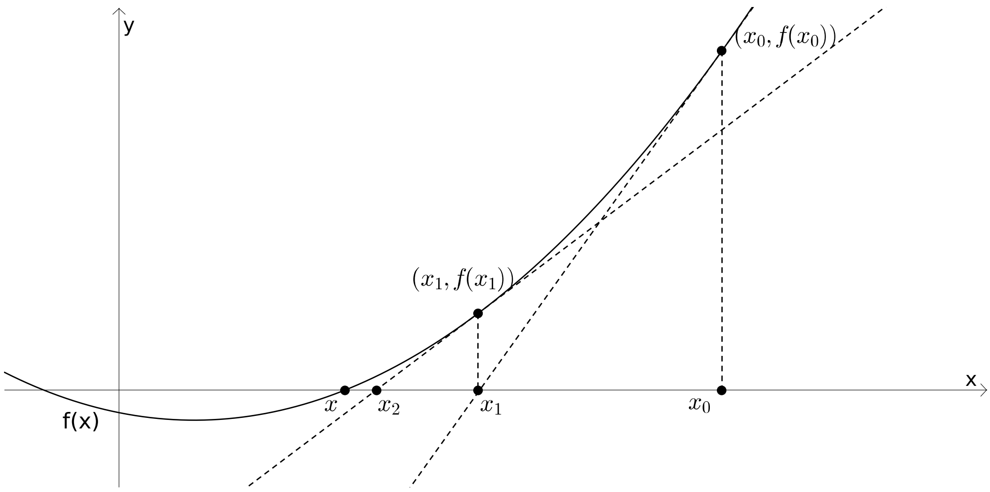

FIGURE 11. Using the Newton-Raphson method to solve an equation $f(x) = 0$.

Warning 3.10.1. Above I said 'if we are lucky', since other things can happen. The iterations could tend to a solution we don't want or they could tend to infinity. In this course I will just ask you to perform so many iterations but please do be aware that things don't always work out nicely in practice and there is a huge area of Mathematics that studies iterative methods like this - such as the area that studies fractals like the Mandelbrot set.

Hear is an example to show how the method works.

Example 3.10.2. Starting with the initial guess $x_0 = 0$, apply two iterations of the Newton-Raphson method to obtain an approximate solution of the equation $2x^3 - 9x^2 + 12x + 5 = 0$.

We can write the equation $2x^3 - 9x^2 + 12x + 5 = 0$ as $f(x) = 0$ where, $f(x) = 2x^3 - 9x^2 + 12x + 5$. So $f'(x) = 6x^2 - 18x + 12$ and (1) becomes

$$x_{n+1} = x_n - \frac{2x_n^3 - 9x_n^2 + 12x_n + 5}{6x_n^2 - 18x_n + 12}$$

Using $x_0 = 0$ we obtain

$$x_1 = x_0 - \frac{2x_0^3 - 9x_0^2 + 12x_0 + 5}{6x_0^2 - 18x_0 + 12} = 0 - \frac{2(0^3) - 9(0^2) + 12(0) + 5}{6(0^2) - 18(0) + 12} = -\frac{5}{12}$$

and then

$$\begin{aligned} x_2 &= x_1 - \frac{2x_1^3 - 9x_1^2 + 12x_1 + 5}{6x_1^2 - 18x_1 + 12} \\ &= -\frac{5}{12} - \frac{2\left(-\frac{5}{12}\right)^3 - 9\left(-\frac{5}{12}\right)^2 + 12\left(-\frac{5}{12}\right) + 5}{6\left(-\frac{5}{12}\right)^2 - 18\left(-\frac{5}{12}\right) + 12} \\ &= -\frac{1480}{4437}. \end{aligned}$$

Note that to three decimal places $-\frac{1480}{4437} \simeq -0.334$, which is pretty close to the actual solution -0.329 (also to three decimal places).