Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Assignment 1 Solutions

1. (a) $2\begin{pmatrix} 2 & 5 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 10 \\ -6 & 0 \end{pmatrix}$

   (b) $\frac{1}{5}\begin{pmatrix} -6 & -3 \\ 3 & 7 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} & -\frac{3}{5} \\ \frac{3}{5} & \frac{7}{5} \end{pmatrix}$

   (c) $-\frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{pmatrix}$

2. (a) (i) $a_{32} = -2$

   (ii) $d_{12} = 4$

   (iii) $e_{21} = -2$

   (iv) $f_{31} = -1$

   (b) A is a 3×2 matrix, B is a 2×2 matrix, C is a 1×3 matrix, D is a 3×2 matrix, E is a 2×2 matrix and F is a 3×1 matrix.

   (c) (i) $3A + D$, $2B^T - 3E^T$, $D^T - 5A^T$, $2C^T - 3F$, $2C - 3F^T$, $3F - 2C^T$ and $3F^T - 2C$ can be formed.

   (ii)

$$\begin{aligned}
3A + D &= 3\begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -6 \\ -6 & -9 \\ 9 & -6 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -2 \\ -3 & -12 \\ 13 & -4 \end{pmatrix}
\end{aligned}$$

$$2B^T - 3E^T = 2\begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}^T - 3\begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}^T$$
$$= 2\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} - 3\begin{pmatrix} 4 & -2 \\ -3 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 4 \\ 6 & -8 \end{pmatrix} + \begin{pmatrix} -12 & 6 \\ 9 & -9 \end{pmatrix}$$
$$= \begin{pmatrix} -14 & 10 \\ 15 & -17 \end{pmatrix}$$

$$D^T - 5A^T = \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}^T - 5\begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix}^T$$
$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & -3 & 2 \end{pmatrix} - 5\begin{pmatrix} 1 & -2 & 3 \\ -2 & -3 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & -3 & 2 \end{pmatrix} + \begin{pmatrix} -5 & 10 & -15 \\ 10 & 15 & 10 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & 13 & -11 \\ 14 & 12 & 12 \end{pmatrix}$$

$$2C^T - 3F = 2\begin{pmatrix} -1 & 0 & 9 \end{pmatrix}^T - 3\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$
$$= 2\begin{pmatrix} -1 \\ 0 \\ 9 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$$
$$= \begin{pmatrix} -2 \\ 0 \\ 18 \end{pmatrix} + \begin{pmatrix} 9 \\ -6 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 7 \\ -6 \\ 21 \end{pmatrix}$$

$$2C - 3F^T = \left(2C^T - 3F\right)^T = \begin{pmatrix} 7 \\ -6 \\ 21 \end{pmatrix}^T = \begin{pmatrix} 7 & -6 & 21 \end{pmatrix}$$

$$3F - 2C^T = -\left(2C^T - 3F\right) = -\begin{pmatrix} 7 \\ -6 \\ 21 \end{pmatrix} = \begin{pmatrix} -7 \\ 6 \\ -21 \end{pmatrix}$$

$$3F^T - 2C = -\left(2C - 3F^T\right) = -\begin{pmatrix} 7 & -6 & 21 \end{pmatrix} = \begin{pmatrix} -7 & 6 & -21 \end{pmatrix}$$

(d) (i) AB, AB^T, ED^T, DE, DE^T, CD, D^TC^T, ABD^T, D^TF, $(B^T)^2$ and $(B^2)^T$ are defined.

(ii) AB is a 3×2 matrix, AB^T is a 3×2 matrix, ED^T is a 2×3 matrix, DE is a 3×2 matrix, DE^T is a 3×2 matrix, CD is a 1×2 matrix, D^TC^T is a 2×1 matrix, ABD^T is a 3×3 matrix, D^TF is a 2×1 matrix, $(B^T)^2$ is a 2×2 matrix and $(B^2)^T$ is a 2×2 matrix.

(iii)

$$AB = \begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} -5 & 11 \\ -4 & 6 \\ -7 & 17 \end{pmatrix}$$

$$\begin{aligned} AB^T &= \begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -7 & 10 \\ -7 & 8 \\ -9 & 14 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} ED^T &= \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}^T \\ &= \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 4 & -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -4 & 21 & 10 \\ 8 & -15 & -2 \end{pmatrix} \end{aligned}$$

$$DE = \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ 18 & -18 \\ 12 & -6 \end{pmatrix}$$

$$DE^T = \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -2 & 3 \end{pmatrix}^T$$
$$= \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -3 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 8 \\ 21 & -15 \\ 10 & -2 \end{pmatrix}$$

$$CD = \begin{pmatrix} -1 & 0 & 9 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 34 & 14 \end{pmatrix}$$

$$D^T C^T = (CD)^T = \begin{pmatrix} 34 & 14 \end{pmatrix}^T = \begin{pmatrix} 34 \\ 14 \end{pmatrix}$$

Note that $(AB)^T$ is always equal to $B^T A^T$ whenever the product AB is defined.

$$ABD^T = \begin{pmatrix} 1 & -2 \\ -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}^T$$
$$= \begin{pmatrix} -5 & 11 \\ -4 & 6 \\ -7 & 17 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}^T$$
$$= \begin{pmatrix} -5 & 11 \\ -4 & 6 \\ -7 & 17 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 \\ 4 & -3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 34 & -48 & 2 \\ 16 & -30 & -4 \\ 54 & -72 & 6 \end{pmatrix}$$

$$D^T F = \begin{pmatrix} 2 & 4 \\ 3 & -3 \\ 4 & 2 \end{pmatrix}^T \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & -3 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -20 \end{pmatrix}$$

$$
\begin{aligned}
(B^T)^2 &= B^T B^T \\
&= \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}^T \begin{pmatrix} -1 & 3 \\ 2 & -4 \end{pmatrix}^T \\
&= \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix} \\
&= \begin{pmatrix} 7 & -10 \\ -15 & 22 \end{pmatrix}
\end{aligned}
$$

$$
(B^2)^T = (BB)^T = B^T B^T = (B^T)^2 = \begin{pmatrix} 7 & -10 \\ -15 & 22 \end{pmatrix}
$$

3. (a) $\det(A) = (2)(3) - (4)(1) = 2 \neq 0$, so A has an inverse.
$\det(B) = (-1)(1) - (0)(0) = -1 \neq 0$, so B has an inverse.
$\det(C) = (2)(-4) - (3)(-2) = -2 \neq 0$, so C has an inverse.
$\det(D) = (4)(3) - (-4)(-3) = 0$, so D does not have an inverse.

(b)

$$
A^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}
$$

$$
B^{-1} = \frac{1}{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

Note in this case we have $B^{-1} = B$.

$$
C^{-1} = \frac{1}{-2} \begin{pmatrix} -4 & -3 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} \\ -1 & -1 \end{pmatrix}
$$

4. (a) We will row reduce the augmented matrix $\begin{pmatrix} 1 & -2 & 3 & 14 \\ -1 & 2 & -2 & -3 \\ 1 & -3 & 5 & 5 \end{pmatrix}$

$$\begin{aligned} R2 &\to R2 + R1 \\ R3 &\to R3 - R1 \end{aligned} \quad \begin{pmatrix} 1 & -2 & 3 & 14 \\ 0 & 0 & 1 & 11 \\ 0 & -1 & 2 & -9 \end{pmatrix}$$

$$R2 \leftrightarrow R3 \quad \begin{pmatrix} 1 & -2 & 3 & 14 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

$$R2 \to -R2 \quad \begin{pmatrix} 1 & -2 & 3 & 14 \\ 0 & 1 & -2 & 9 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

$$\begin{aligned} R1 &\to R1 - 3R3 \\ R2 &\to R2 + 2R3 \end{aligned} \quad \begin{pmatrix} 1 & -2 & 0 & -19 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

$$R1 \to R1 + 2R2 \quad \begin{pmatrix} 1 & 0 & 0 & 43 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 11 \end{pmatrix}$$

Hence the solution is $x = 43$, $y = 31$, $z = 11$.

(b) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 2 & -1 & 5 \\ -1 & 3 & -4 & 5 \\ 3 & -1 & 4 & 1 \end{pmatrix}$

$$\begin{aligned} R2 &\to R2 + R1 \\ R3 &\to R3 - 3R1 \end{aligned} \quad \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 5 & -5 & 10 \\ 0 & -7 & 7 & -14 \end{pmatrix}$$

$$R2 \to \tfrac{1}{5}R2 \quad \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & -7 & 7 & -14 \end{pmatrix}$$

$$R3 \to R3 + 7R2 \quad \begin{pmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \to R1 - 2R2 \quad \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since we have a row of zeros across the bottom, we have a free variable and since there is no leading 1 in the z column, z is the free variable. Thus we can let $z = t$, $t \in \mathbb{R}$. From the top row we then have $x+z = 1$, so $x = 1-z = 1-t$. Finally from the second row we have $y - z = 2$, so $y = 2 + z = 2 + t$. Thus the solution is $x = 1 - t$, $y = 2 + t$, $z = t$ for any real number t.

(c) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 1 & -2 & 4 \\ 3 & 5 & -6 & 6 \\ 2 & -3 & -4 & 5 \end{pmatrix}$

$$\begin{array}{rcl} R2 & \to & R2 - 3R1 \\ R3 & \to & R3 - 2R1 \end{array} \quad \begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & 2 & 0 & -6 \\ 0 & -5 & 0 & -3 \end{pmatrix}$$

$$R2 \to \tfrac{1}{2}R2 \quad \begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & -5 & 0 & -3 \end{pmatrix}$$

$$R3 \to R3 + 5R2 \quad \begin{pmatrix} 1 & 1 & -2 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & -18 \end{pmatrix}$$

Now, the system of equations corresponding to this augmented matrix has as its third equation $0x + 0y + 0z = -18$. For all values of x, y and z, this reduces to $0 = -18$ which is clearly impossible. Thus the system of equations has no solutions.

5. (a) We will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{rcl} R2 & \to & R2 + R1 \\ R3 & \to & R3 - R1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$R2 \to -R2 \quad \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + R2 \quad \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right)$$

$$R3 \to -R3 \quad \left(\begin{array}{rrr|rrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)$$

$$R1 \to R1 - R3 \quad \left(\begin{array}{rrr|rrr} 1 & -2 & 0 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)$$

$$R1 \to R1 + 2R2 \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & -3 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & -3 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$.

(b) Again we will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & -4 & -3 & 0 & 1 & 0 \\ 0 & -12 & -12 & 0 & 0 & 1 \end{array}\right)$.

$$R2 \to R2 - R1 \quad \left(\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -6 & -6 & -1 & 1 & 0 \\ 0 & -12 & -12 & 0 & 0 & 1 \end{array}\right)$$

$$R2 \to -\tfrac{1}{6}R2 \quad \left(\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & -12 & -12 & 0 & 0 & 1 \end{array}\right)$$

$$R2 \to R3 + 12R2 \quad \left(\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 2 & -2 & 1 \end{array}\right)$$

At this stage we have three zeros along the bottom row of the matrix on the left. This means that there is no way we can end up with an identity matrix on the left and so $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -4 & -3 \\ 0 & -12 & -12 \end{pmatrix}$ is not invertible.

6. (a)

$$\det \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 1 & -3 & 0 \end{pmatrix} = 1 \begin{vmatrix} 1 & -1 \\ -3 & 0 \end{vmatrix} - (-2) \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix}$$
$$\begin{aligned} &=1(1 \times 0 - (-1) \times (-3)) \\ &\quad + 2(-1 \times 0 - (-1) \times 1) \\ &\quad + 1(-1 \times (-3) - 1 \times 1) \\ &=1(-3) + 2(1) + 1(2) \\ &= - 3 + 2 + 2 \\ &=1. \end{aligned}$$

Since the determinant is non-zero, we have confirmed that A is invertible.

(b)

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & -4 & -3 \\ 0 & -12 & -12 \end{pmatrix} = 1 \begin{vmatrix} -4 & -3 \\ -12 & -12 \end{vmatrix} - 2 \begin{vmatrix} 1 & -3 \\ 0 & -12 \end{vmatrix} + 3 \begin{vmatrix} 1 & -4 \\ 0 & -12 \end{vmatrix}$$
$$\begin{aligned} &=1(-4 \times (-12) - (-3) \times (-12)) \\ &\quad - 2(-1 \times (-12) - (-3) \times 0) \\ &\quad + 3(1 \times (-12) - (-4) \times 0) \\ &=1(12) - 2(-12) + 3(-12) \\ &=12 + 24 - 36 \\ &=0. \end{aligned}$$

Since the determinant is zero, we have confirmed that B is not invertible.

7. (a) We have $\|(2, 3)\| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$.
Hence a unit vector in the direction of $(2, 3)$ is $\left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$.

(b) We have $\|(2, -3)\| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$.
Hence a unit vector in the direction of $(2, 3)$ is $\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\right)$.

(c) We have $\|(1, 1, 1)\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{1 + 1 + 1} = \sqrt{3}$.
Hence a unit vector in the direction of $(1, 1, 1)$ is $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

(d) We have
$\|(0, 0, -4, 5)\| = \sqrt{0^2 + 0^2 + (-4)^2 + 5^2} = \sqrt{0 + 0 + 16 + 25} = \sqrt{41}$.
Hence a unit vector in the direction of $(0, 0, -4, 5)$ is $\left(0, 0, -\frac{4}{\sqrt{41}}, \frac{5}{\sqrt{41}}\right)$.

8. In this question we will use the formula $\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}\right)$, where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

(a) The angle between $(1, 3)$ and $(3, 1)$ is

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{(1,3)\cdot(3,1)}{\|(1,3)\|\cdot\|(3,1)\|}\right)\\
&= \cos^{-1}\left(\frac{(1)(3)+(3)(1)}{\sqrt{1^2+3^2}\cdot\sqrt{1^2+3^2}}\right)\\
&= \cos^{-1}\left(\frac{6}{\sqrt{10}\cdot\sqrt{10}}\right)\\
&= \cos^{-1}\left(\frac{6}{10}\right)\\
&\simeq 0.93 \text{ to 2 d.p.}
\end{aligned}$$

(b) The angle between $(1, 2)$ and $(2, -1)$ is

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{(1,2)\cdot(2,-1)}{\|(1,2)\|\cdot\|(2,-1)\|}\right)\\
&= \cos^{-1}\left(\frac{(1)(2)+(2)(-1)}{\sqrt{1^2+2^2}\cdot\sqrt{2^2+(-1)^2}}\right)\\
&= \cos^{-1}\left(\frac{0}{\sqrt{1^2+2^2}\cdot\sqrt{2^2+(-1)^2}}\right)\\
&= \cos^{-1}(0)\\
&= \frac{\pi}{2}\\
&\simeq 1.57 \text{ to 2 d.p.}
\end{aligned}$$

(c) The angle between $(1,0,1)$ and $(2,2,2)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(1,0,1)\cdot(2,2,2)}{\|(1,0,1)\|\cdot\|(2,2,2)\|}\right)\\ &= \cos^{-1}\left(\frac{(1)(2)+(0)(2)+(1)(2)}{\sqrt{1^2+0^2+1^2}\cdot\sqrt{2^2+2^2+2^2}}\right)\\ &= \cos^{-1}\left(\frac{4}{\sqrt{2}\cdot\sqrt{12}}\right)\\ &= \cos^{-1}\left(\frac{4}{\sqrt{24}}\right)\\ &\simeq 0.62 \text{ to 2 d.p.}\end{aligned}$$

(d) Th angle between $(-1,-2,-1)$ and $(1,2,2)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(-1,-2,-1)\cdot(1,2,2)}{\|(-1,-2,-1)\|\cdot\|(1,2,2)\|}\right)\\ &= \cos^{-1}\left(\frac{(-1)(1)+(-2)(2)+(-1)(2)}{\sqrt{(-1)^2+(-2)^2+(-1)^2}\cdot\sqrt{1^2+2^2+2^2}}\right)\\ &= \cos^{-1}\left(\frac{-7}{\sqrt{6}\cdot\sqrt{9}}\right)\\ &= \cos^{-1}\left(\frac{-7}{\sqrt{54}}\right)\\ &\simeq 2.83 \text{ to 2 d.p.}\end{aligned}$$

9. (a)

$$\begin{aligned}(1,2,3)\times(-3,-2,-1) &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -3 & -2 & -1\end{vmatrix}\\ &= \begin{vmatrix}2 & 3\\ -2 & -1\end{vmatrix}\hat{i} - \begin{vmatrix}1 & 3\\ -3 & -1\end{vmatrix}\hat{j} + \begin{vmatrix}1 & 2\\ -3 & -2\end{vmatrix}\hat{k}\\ &=(2\times(-1)-3\times(-2))\hat{i}-(1\times(-1)-3\times(-3))\hat{j}\\ &\quad+(1\times(-2)-2\times(-3))\hat{k}\\ &=4\hat{i}-8\hat{j}+4\hat{k}.\end{aligned}$$

(b)

$$\begin{aligned}(0,0,1)\times(0,1,0) &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ 0 & 1 & 0\end{vmatrix}\\ &= \begin{vmatrix}0 & 1\\ 1 & 0\end{vmatrix}\hat{i} - \begin{vmatrix}0 & 1\\ 0 & 0\end{vmatrix}\hat{j} + \begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}\hat{k}\\ &= (0\times 0-1\times 1)\hat{i}-(0\times 0-1\times 0)\hat{j}+(0\times 1-0\times 0)\hat{k}\\ &= -\hat{i}+0\hat{j}+0\hat{k} = -\hat{i}.\end{aligned}$$

(c)

$$
\begin{aligned}
(2,1,-3)\times(4,2,-6) &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -3 \\ 4 & 2 & -6 \end{vmatrix} \\
&= \begin{vmatrix} 1 & -3 \\ 2 & -6 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix}\hat{k} \\
&=(1\times(-6)-(-3)\times 2)\hat{\imath} - (2\times(-6)-(-3)\times 4)\hat{\jmath} \\
&\quad + (2\times 2 - 1\times 4)\hat{k} \\
&=0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = \mathbf{0}.
\end{aligned}
$$

Note we end up with the zero vector since the original vectors are parallel.

10. (a) We have

$$
\begin{aligned}
\det\left[\begin{pmatrix} 13 & 14 \\ -7 & -8 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} 13-\lambda & 14 \\ -7 & -8-\lambda \end{pmatrix} \\
&= (13-\lambda)(-8-\lambda) - 14(-7) \\
&= \lambda^2 - 5\lambda - 104 + 98 \\
&= \lambda^2 - 5\lambda - 6
\end{aligned}
$$

Hence the characteristic equation is $\lambda^2 - 5\lambda - 6 = 0$ or $(\lambda-6)(\lambda+1) = 0$. Thus the eigenvalues are $\lambda = 6$ and $\lambda = -1$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = 6$:
We have $\begin{pmatrix} 13 & 14 \\ -7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 13x+14y \\ -7x-8y \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \end{pmatrix}$.
Hence we have the equations $13x+14y = 6x$ and $-7x-8y = 6y$. Both these equations reduce to $x = -2y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$\lambda = -1$:
We have $\begin{pmatrix} 13 & 14 \\ -7 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 13x+14y \\ -7x-8y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.
Hence we have the equations $13x+14y = -x$ and $-7x-8y = -y$. Both these equations reduce to $x = -y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) We have

$$\det\left[\begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 2-\lambda & -8 \\ 1 & -4-\lambda \end{pmatrix}$$
$$= (2-\lambda)(-4-\lambda) - (-8)(1)$$
$$= \lambda^2 + 2\lambda - 12 + 8$$
$$= \lambda^2 + 2\lambda$$

Hence the characteristic equation is $\lambda^2 + 2\lambda = 0$ or $\lambda(\lambda + 2) = 0$.
Thus the eigenvalues are $\lambda = 0$ and $\lambda = -2$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = 0$:
We have $\begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 2x-8y \\ x-4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence we have the equations $2x - 8y = 0$ and $x - 4y = 0$. Both these equations reduce to $x = 4y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

$\lambda = -2$:
We have $\begin{pmatrix} 2 & -8 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 2x-8y \\ x-4y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$.
Hence we have the equations $2x - 8y = -2x$ and $x - 4y = -2y$. Both these equations reduce to $x = 2y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(c) We have

$$\det\left[\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix}$$
$$= (4-\lambda)(2-\lambda) - 1(-1)$$
$$= \lambda^2 - 6\lambda + 8 + 1$$
$$= \lambda^2 - 6\lambda + 9$$

Hence the characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$ or $(\lambda - 3)^2 = 0$.
Thus there is only one eigenvalue $\lambda = 3$.
We will now find the eigenvector(s) corresponding to this eigenvalue.
We have $\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 4x+y \\ -x+2y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$.
Hence we have the equations $4x + y = 3x$ and $-x + 2y = 3y$. Both these equations reduce to $x = -y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Note there is only one eigenvector so this matrix has a defect.

(d) We have

$$\det\left[\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} -3-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$$
$$= (-3-\lambda)(4-\lambda) - 0(0)$$
$$= (-3-\lambda)(4-\lambda).$$

Hence the characteristic equation is $(-3-\lambda)(4-\lambda) = 0$ and the eigenvalues are $\lambda = -3$ and $\lambda = 4$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -3$:
We have $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -3x \\ 4y \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$. Hence we have the equations $-3x = -3x$ and $4y = -3y$. The first of these equations places no restriction on x and the second yields $y = 0$. Thus taking $x = 1$, say, we obtain the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\lambda = 4$:
We have $\begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -3x \\ 4y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$. Hence we have the equations $-3x = 4x$ and $4y = 4y$. The first of these equations yields $x = 0$ and the second places no restriction on y. Thus taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.