Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Assignment 2 Solutions

1. (a)

$$|1+3i| = \sqrt{1^2+3^2} = \sqrt{1+9} = \sqrt{10},$$
$$\overline{1+3i} = 1-3i, \quad \mathrm{Re}(1+3i) = 1, \quad \mathrm{Im}(1+3i) = 3.$$

$$|2+i| = \sqrt{2^2+1^2} = \sqrt{4+1} = \sqrt{5},$$
$$\overline{2+i} = 2-i, \quad \mathrm{Re}(2+i) = 2, \quad \mathrm{Im}(2+i) = 1.$$

$$(1+3i)+(2+i) = (1+2)+(3+1)i = 3+4i$$
$$(1+3i)-(2+i) = (1-2)+(3-1)i = -1+2i$$
$$(1+3i)(2+i) = ((1)(2)-(3)(1)) + ((1)(1)+(3)(2))i = -1+7i$$
$$\frac{1+3i}{2+i} = \frac{1+3i}{2+i}\cdot\frac{2-i}{2-i} = \frac{5+5i}{5} = 1+i$$
$$\frac{2+i}{1+3i} = \frac{2+i}{1+3i}\cdot\frac{1-3i}{1-3i} = \frac{5-5i}{10} = \frac{1}{2}-\frac{1}{2}i$$

(b)

$$|1-2i| = \sqrt{1^2+(-2)^2} = \sqrt{1+4} = \sqrt{5},$$
$$\overline{1-2i} = 1+2i, \quad \mathrm{Re}(1-2i) = 1, \quad \mathrm{Im}(1-2i) = -2.$$

$$|-2-i| = \sqrt{(-2)^2+(-1)^2} = \sqrt{4+1} = \sqrt{5},$$
$$\overline{-2-i} = -2+i, \quad \mathrm{Re}(-2-i) = -2, \quad \mathrm{Im}(-2-i) = -1.$$

$$(1-2i)+(-2-i) = (1-2)+(-2-1)i = -1-3i$$
$$(1-2i)-(-2-i) = (1+2)+(-2+1)i = 3-i$$
$$(1-2i)(-2-i) = ((1)(-2)-(-2)(-1)) + ((1)(-1)+(-2)(-2))i = -4+3i$$
$$\frac{1-2i}{-2-i} = \frac{1-2i}{-2-i}\cdot\frac{-2+i}{-2+i} = \frac{0+5i}{5} = i$$
$$\frac{-2-i}{1-2i} = \frac{-2-i}{1-2i}\cdot\frac{1+2i}{1+2i} = \frac{0-5i}{5} = -i$$

(c)

$$|2i| = \sqrt{0^2+2^2} = \sqrt{4} = 2,$$
$$\overline{2i} = -2i, \quad \mathrm{Re}(2i) = 0, \quad \mathrm{Im}(2i) = 2.$$

$$
\begin{aligned}
|-4| &= \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4, \\
\overline{-4} &= -4, \quad \text{Re}(-4) = -4, \quad \text{Im}(-4) = 0.
\end{aligned}
$$

$$
\begin{aligned}
(2i) + (-4) &= -4 + 2i \\
(2i) - (-4) &= 4 + 2i \\
(2i)(-4) &= -8i \\
\frac{2i}{-4} &= -\frac{1}{2}i \\
\frac{-4}{2i} &= \frac{-4}{2i} \cdot \frac{-2i}{-2i} = \frac{8i}{4} = 2i
\end{aligned}
$$

(d)

$$
\begin{aligned}
|2 - 4i| &= \sqrt{2^2 + (-4)^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}, \\
\overline{2 - 4i} &= 2 + 4i, \quad \text{Re}(2 - 4i) = 2, \quad \text{Im}(2 - 4i) = -4.
\end{aligned}
$$

$$
\begin{aligned}
|-1 + 6i| &= \sqrt{(-1)^2 + 6^2} = \sqrt{1 + 36} = \sqrt{37}, \\
\overline{-1 + 6i} &= -1 - 6i, \quad \text{Re}(-1 + 6i) = -1, \quad \text{Im}(-1 + 6i) = 6.
\end{aligned}
$$

$$
\begin{aligned}
(2 - 4i) + (-1 + 6i) &= (2 - 1) + (-4 + 6)i = 1 + 2i \\
(2 - 4i) - (-1 + 6i) &= (2 + 1) + (-4 - 6)i = 3 - 10i \\
(2 - 4i)(-1 + 6i) &= ((2)(-1) - (-4)(6)) + ((2)(6) + (-4)(-1))i = 22 + 16i \\
\frac{2 - 4i}{-1 + 6i} &= \frac{2 - 4i}{-1 + 6i} \cdot \frac{-1 - 6i}{-1 - 6i} = \frac{-26 - 8i}{37} = -\frac{26}{37} - \frac{8}{37}i \\
\frac{-1 + 6i}{2 - 4i} &= \frac{-1 + 6i}{2 - 4i} \cdot \frac{2 + 4i}{2 + 4i} = \frac{-26 + 8i}{20} = -\frac{13}{10} + \frac{2}{5}i
\end{aligned}
$$

2. (a) Since the real and the imaginary parts of $\sqrt{3} + i$ are both positive, the argument, θ, of $\sqrt{3} + i$ is given by

$$
\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.
$$

Also, the magnitude, r of $\sqrt{3} + i$ is $r = \sqrt{\left(\sqrt{3} = 1^2\right)^2} = \sqrt{3 + 1} = \sqrt{4} = 2$. Hence, $\sqrt{3} + i$ in polar form is

$$
\sqrt{3} + i = 2\left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right).
$$

(b) Since $2i$ lies on the positive imaginary axis, we can immediately say that its argument is $\theta = \frac{\pi}{2}$. Since its magnitude (i.e., its distance from the origin) is 2, its polar form is

$$
2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)\right).
$$

(c) The real part of $-1+i$ is negative and its imaginary part is positive, so we are in the situation of Figure 6 in the Complex Numbers notes. Hence the argument of $-1+i$ is

$$\theta = \pi - \phi = \pi - \tan^{-1}\left(\left|\frac{1}{-1}\right|\right) = \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Also, the magnitude of $-1+i$ is $r = \sqrt{(-1)^2+1^2} = \sqrt{1+1} = \sqrt{2}$. Hence, $-1+i$ in polar form is

$$-1+i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

(d) Since -3 lies on the negative real axis, we can immediately say that its argument is $\theta = \pi$. Since its magnitude (i.e., its distance from the origin) is 3, its polar form is

$$-3 = 3\left(\cos(\pi) + i\sin(\pi)\right).$$

(e) The real and imaginary parts of $-2 - \frac{2}{\sqrt{3}}i$ are both negative, so we are in the situation of Figure 7 in the Complex Numbers notes. Hence the argument of $-2 - \frac{2}{\sqrt{3}}i$ is

$$\theta = \phi - \pi = \tan^{-1}\left(\left|\frac{2/\sqrt{3}}{-2}\right|\right) - \pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$$

Also, the magnitude of $-2 - \frac{2}{\sqrt{3}}i$ is

$$r = \sqrt{(-2)^2 + \left(-\frac{2}{\sqrt{3}}\right)^2} = \sqrt{4+\frac{4}{3}} = \sqrt{\frac{16}{3}} = \frac{4}{\sqrt{3}}.$$

Hence, $-2 - \frac{2}{\sqrt{3}}i$ in polar form is

$$-2 - \frac{2}{\sqrt{3}}i = \frac{4}{\sqrt{3}}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right).$$

(f) The real part of $3 - 3\sqrt{3}i$ is positive and its imaginary part is negative, so we are in the situation of Figure 5 in the Complex Numbers notes. Hence the argument of $3 - 3\sqrt{3}i$ is

$$\theta = -\phi = -\tan^{-1}\left(\left|\frac{-3\sqrt{3}}{3}\right|\right) = -\tan^{-1}\left(\sqrt{3}\right) = -\frac{\pi}{3}.$$

Also, the magnitude of $3 - 3\sqrt{3}i$ is

$$r = \sqrt{(-3)^2 + \left(3\sqrt{3}\right)^2} = \sqrt{9+27} = \sqrt{36} = 6.$$

Hence, $3 - 3\sqrt{3}i$ in polar form is

$$3 - 3\sqrt{3}i = 6\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right).$$

3. In all these problems we will use Corollary 2.3.9 from the Complex Numbers notes. That is we will use

$$(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

(a)

$$\begin{aligned}
(\sqrt{3}+i)^2 &= \left(2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right)^2 \\
&= 2^2\left(\cos\left(\frac{2\pi}{6}\right) + i\sin\left(\frac{2\pi}{6}\right)\right) \\
&= 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right) \\
&= 2 + 2\sqrt{3}i \\
&\simeq 2 + 3.464i.
\end{aligned}$$

(b)

$$\begin{aligned}
(2i)^3 &= \left(2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)\right)^3 \\
&= 2^3\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) \\
&= 8\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) \\
&= 0 - 8i \\
&= -8i.
\end{aligned}$$

(c)

$$\begin{aligned}
(-1+i)^4 &= \left(\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right)^4 \\
&= (\sqrt{2})^4\left(\cos(3\pi) + i\sin(3\pi)\right) \\
&= 4\left(\cos(3\pi) + i\sin(3\pi)\right) \\
&= -4 + 0i \\
&= -4.
\end{aligned}$$

(d)

$$\begin{aligned}
(-3)^5 &= (3(\cos(\pi) + i\sin(\pi)))^5 \\
&= 3^5(\cos(5\pi) + i\sin(5\pi)) \\
&= 243(\cos(5\pi) + i\sin(5\pi)) \\
&= -243 + 0i \\
&= -243.
\end{aligned}$$

(e)

$$\begin{aligned}\left(-2-\frac{2}{\sqrt{3}}\right)^6 &= \left(\frac{4}{\sqrt{3}}\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)\right)^6\\ &= \left(\frac{4}{\sqrt{3}}\right)^6(\cos(-5\pi)+i\sin(-5\pi))\\ &= \frac{4096}{27}(\cos(-5\pi)+i\sin(-5\pi))\\ &= -\frac{4096}{27}+0i\\ &\simeq -151.704.\end{aligned}$$

(f)

$$\begin{aligned}\left(3-3\sqrt{3}i\right)^7 &= \left(6\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)\right)^7\\ &= 6^7\left(\cos\left(-\frac{7\pi}{3}\right)+i\sin\left(-\frac{7\pi}{3}\right)\right)\\ &= 279936\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)\\ &= 139968-139968\sqrt{3}i\\ &\simeq 139968-242431.687i.\end{aligned}$$

4. In all these problems we will we will use the fact (see P.12 of the Complex Numbers notes) that the nth roots are given by

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right) \quad k=0,1,\ldots,n-1.$$

(a) In this case $\sqrt{3}+i = 2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$ and we are looking for the square roots, so we take $n=2$. Thus the roots are

$$z_k = 2^{\frac{1}{2}}\left(\cos\left(\frac{\pi/6}{2}+\frac{2k\pi}{2}\right)+i\sin\left(\frac{\pi/6}{2}+\frac{2k\pi}{2}\right)\right) \quad k=0,1.$$

That is

$$\begin{aligned}z_0 &= 2^{\frac{1}{2}}\left(\cos\left(\frac{\pi}{12}\right)+i\sin\left(\frac{\pi}{12}\right)\right) \simeq 1.366+0.366i\\ z_1 &= 2^{\frac{1}{2}}\left(\cos\left(\frac{13\pi}{12}\right)+i\sin\left(\frac{13\pi}{12}\right)\right) \simeq -1.366-0.366i\end{aligned}$$

(b) In this case $2i = 2\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$ and we are looking for the third roots, so we take $n=3$. Thus the roots are

$$z_k = 2^{\frac{1}{3}}\left(\cos\left(\frac{\pi/2}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\pi/2}{3}+\frac{2k\pi}{3}\right)\right) \quad k=0,1,2.$$

That is

$$z_0 = 2^{\frac{1}{3}}\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) \simeq 1.091 + 0.630i$$
$$z_1 = 2^{\frac{1}{3}}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \simeq -1.091 + 0.630i$$
$$z_2 = 2^{\frac{1}{3}}\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right) \simeq -1.260i$$

(c) In this case $-1+i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$ and we are looking for the fourth roots, so we take $n = 4$. Thus the roots are

$$z_k = (\sqrt{2})^{\frac{1}{4}}\left(\cos\left(\frac{3\pi/4}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{3\pi/4}{4} + \frac{2k\pi}{4}\right)\right) \qquad k = 0, 1, 2, 3.$$

That is

$$z_0 = 2^{\frac{1}{8}}\left(\cos\left(\frac{3\pi}{16}\right) + i\sin\left(\frac{3\pi}{16}\right)\right) \simeq 0.907 + 0.606i$$
$$z_1 = 2^{\frac{1}{8}}\left(\cos\left(\frac{11\pi}{16}\right) + i\sin\left(\frac{11\pi}{16}\right)\right) \simeq -0.606 + 0.907i$$
$$z_2 = 2^{\frac{1}{8}}\left(\cos\left(\frac{19\pi}{16}\right) + i\sin\left(\frac{19\pi}{16}\right)\right) \simeq -0.907 - 0.606i$$
$$z_3 = 2^{\frac{1}{8}}\left(\cos\left(\frac{27\pi}{16}\right) + i\sin\left(\frac{27\pi}{16}\right)\right) \simeq 0.606 - 0.907i$$

(d) In this case $-3 = 3(\cos(\pi) + i\sin(\pi))$ and we are looking for the fourth roots, so we take $n = 4$. Thus the roots are

$$z_k = 3^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right)\right) \qquad k = 0, 1, 2, 3.$$

That is

$$z_0 = 3^{\frac{1}{4}}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \simeq 0.931 + 0.931i$$
$$z_1 = 3^{\frac{1}{4}}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right) \simeq -0.931 + 0.931i$$
$$z_2 = 3^{\frac{1}{4}}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) \simeq -0.931 - 0.931i$$
$$z_3 = 3^{\frac{1}{4}}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right) \simeq 0.931 - 0.931i$$

(e) In this case $-2-\frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$ and we are looking for the third roots, so we take $n = 3$. Thus the roots are

$$z_k = \left(\frac{4}{\sqrt{3}}\right)^{\frac{1}{3}}\left(\cos\left(\frac{-5\pi/6}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{-5\pi/6}{3} + \frac{2k\pi}{3}\right)\right) \qquad k = 0, 1, 2.$$

That is

$$
\begin{aligned}
z_0 &= \frac{4^{\frac{1}{3}}}{3^{\frac{1}{6}}}\left(\cos\left(-\frac{5\pi}{18}\right) + i\sin\left(-\frac{5\pi}{18}\right)\right) \simeq 0.850 - 1.013i \\
z_1 &= \frac{4^{\frac{1}{3}}}{3^{\frac{1}{6}}}\left(\cos\left(\frac{7\pi}{18}\right) + i\sin\left(\frac{7\pi}{18}\right)\right) \simeq 0.452 + 1.242i \\
z_2 &= \frac{4^{\frac{1}{3}}}{3^{\frac{1}{6}}}\left(\cos\left(\frac{19\pi}{18}\right) + i\sin\left(\frac{19\pi}{18}\right)\right) \simeq -1.302 - 0.230i
\end{aligned}
$$

(f) In this case $3\sqrt{3} - 3i = 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ and we are looking for the square roots, so we take $n = 2$. Thus the roots are

$$
z_k = 6^{\frac{1}{2}}\left(\cos\left(\frac{-\pi/3}{2} + \frac{2k\pi}{2}\right) + i\sin\left(\frac{-\pi/3}{2} + \frac{2k\pi}{2}\right)\right) \quad k = 0, 1.
$$

That is

$$
\begin{aligned}
z_0 &= 6^{\frac{1}{2}}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) \simeq 2.121 - 1.225i \\
z_1 &= 6^{\frac{1}{2}}\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \simeq -2.121 + 1.225i
\end{aligned}
$$