Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Assignment 3 Solutions

1. (a) Here we have to use the product rule twice. First let us differentiate $g(x) = 4x^3e^{-x}$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(4x^3\right)e^{-x} + 4x^3\frac{d}{dx}\left(e^{-x}\right) \\ &= 12x^2e^{-x} + 4x^3\left(-e^{-x}\right) \\ &= 12x^2e^{-x} - 4x^3e^{-x}. \end{aligned}$$

Next we differentiate $h(x) = \cos(-2x)\ln(-5x)$ (where $x < 0$).

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(\cos(-2x)\right)\ln(-5x) + \cos(-2x)\frac{d}{dx}\left(\ln(-5x)\right) \\ &= 2\sin(-2x)\ln(-5x) + \cos(-2x)\frac{1}{x}. \end{aligned}$$

Putting this together we obtain (for $x < 0$)

$$f'(x) = g'(x) + h'(x) = e^{-x}\left(12x^2 - 4x^3\right) + 2\sin(-2x)\ln(-5x) + \frac{\cos(-2x)}{x}.$$

(b) Again we have to use the product rule twice. First let us differentiate $g(x) = x^{-\frac{1}{2}}\cos(2x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(x^{-\frac{1}{2}}\right)\cos(2x) + x^{-\frac{1}{2}}\frac{d}{dx}\left(\cos(2x)\right) \\ &= -\frac{1}{2}x^{-\frac{3}{2}}\cos(2x) + x^{-\frac{1}{2}}\left(-2\sin(2x)\right) \\ &= -\frac{1}{2}x^{-\frac{3}{2}}\cos(2x) - 2x^{-\frac{1}{2}}\sin(2x). \end{aligned}$$

Next we differentiate $h(x) = -x^3\sin(-4x)$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(-x^3\right)\sin(-4x) + \left(-x^3\right)\frac{d}{dx}\left(\sin(-4x)\right) \\ &= -3x^2\sin(-4x) - x^3(-4\cos(-4x)) \\ &= -3x^2\sin(-4x) + 4x^3\cos(-4x). \end{aligned}$$

Putting this together we obtain

$$\begin{aligned} f'(x) &= g'(x) + h'(x) \\ &= -\frac{1}{2}x^{-\frac{3}{2}}\cos(2x) - 2x^{-\frac{1}{2}}\sin(2x) - 3x^2\sin(-4x) + 4x^3\cos(-4x). \end{aligned}$$

(c) Here we use the quotient rule. If $\cos(3x) \neq 0$, we have

$$\begin{aligned} f'(x) =& \frac{\frac{d}{dx}(\sin(2x))\cos(3x) - \sin(2x)\frac{d}{dx}(\cos(3x))}{\cos^2(3x)} \\ =& \frac{2\cos(2x)\cos(3x) - \sin(2x)(-3\sin(3x))}{\cos^2(3x)} \\ =& \frac{2\cos(2x)\cos(3x) + 3\sin(2x)\sin(3x)}{\cos^2(3x)}. \end{aligned}$$

(d) Again we use the quotient rule. If $x > \frac{1}{3}$, then

$$\begin{aligned} f'(x) =& \frac{\frac{d}{dx}(x^3)\ln(3x) - x^3\frac{d}{dx}(\ln(3x))}{\ln(3x)^2} \\ =& \frac{3x^2\ln(3x) - x^3\left(\frac{1}{x}\right)}{\ln(3x)^2} \\ =& \frac{3x^2\ln(3x) - x^2}{\ln(3x)^2} \\ =& \frac{x^2(3\ln(3x) - 1)}{\ln(3x)^2}. \end{aligned}$$

(e) Here we have a product in the numerator and the denomiator, so we have to use the product rule twice before we use the quotient rule. First let us differentiate $g(x) = x^2\cos(3x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(x^2\right)\cos(3x) + x^2\frac{d}{dx}(\cos(3x)) \\ &= 2x\cos(3x) + x^2(-3\sin(3x)) \\ &= 2x\cos(3x) - 3x^2\sin(3x). \end{aligned}$$

Next we differentiate $h(x) = e^{2x}\sin(x)$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(e^{2x}\right)\sin(x) + e^{2x}\frac{d}{dx}(\sin(x)) \\ &= 2e^{2x}\sin(x) + e^{2x}\cos(x) \\ &= e^{2x}(2\sin(x) + \cos(x)). \end{aligned}$$

We can now use the quotient rule. If $\sin(x) \neq 0$, then

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{(2x\cos(3x) - 3x^2\sin(3x))\,e^{2x}\sin(x) - x^2\cos(3x)e^{2x}(2\sin(x) + \cos(x))}{\left(e^{2x}\sin(x)\right)^2}. \end{aligned}$$

(f) Here we just have a product in the numerator, so let us deal with that first. If $g(x) = e^x(x^3 - 2x^2 + 5)$ then

$$\begin{aligned} g'(x) &= \frac{d}{dx}(e^x)\left(x^3 - 2x^2 + 5\right) + e^x\frac{d}{dx}\left(x^3 - 2x^2 + 5\right) \\ &= e^x\left(x^3 - 2x^2 + 5\right) + e^x\left(3x^2 - 4x\right) \\ &= e^x\left(x^3 + x^2 - 4x + 5\right). \end{aligned}$$

Next, if $h(x) = \cos(-4x)$, then $h'(x) = 4\sin(-4x)$. Hence, if $\cos(-4x) \neq 0$, we can use the quotient rule to obtain

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{e^x\left(x^3 + x^2 - 4x + 5\right)\cos(-4x) - e^x(x^3 - 2x^2 + 5)4\sin(-4x)}{\cos^2(-4x)} \\ &= \frac{e^x\left(x^3 + x^2 - 4x + 5\right)\cos(-4x) - 4e^x(x^3 - 2x^2 + 5)\sin(-4x)}{\cos^2(-4x)}. \end{aligned}$$

(g) Here we will use the chain rule with $u = x^3 - 2x + 5$ and $f(x) = \cos(u)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u)\left(3x^2 - 2\right) = -\sin\left(x^3 - 2x + 5\right)\left(3x^2 - 2\right).$$

(h) Here we will have to use the chain rule twice. Let us first differentiate $u(x) = \sin(\sin(x))$. If we let $v(x) = \sin(x)$, then $u(x) = \sin(v)$. Hence

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \cos(v)\cos(x) = \cos(\sin(x))\cos(x).$$

We are now in a position to differentiate $y = \sin(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u)\cos(\sin(x))\cos(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x).$$

(i) Here we will use the chain rule with $u(x) = xe^x$ and $y(u) = e^u$. However before we can do this, we will have to differentiate $u(x)$ using the product rule.

$$\frac{du}{dx} = \frac{d}{dx}(x)\,e^x + x\frac{d}{dx}\left(e^x\right) = 1e^x + xe^x = e^x\left(1 + x\right).$$

We can now use the chain rule to differentiate $y(x)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u e^x\left(1 + x\right) \\ &= e^{xe^x} e^x\left(1 + x\right) \\ &= e^{xe^x + x}\left(1 + x\right) \\ &= e^{x(e^x + 1)}\left(1 + x\right). \end{aligned}$$

2. In all these questions we will differentiate $f(x)$ and solve the equation $f'(x) = 0$ to find the critical points.

(a) In this case $f'(x) = 3x^2 + 6x - 24$. Thus we have to solve the equation

$$\begin{aligned} 3x^2 + 6x - 24 = 0 &\Leftrightarrow x^2 + 2x - 8 = 0 \\ &\Leftrightarrow (x + 4)(x - 2) = 0 \\ &\Leftrightarrow x = -4 \text{ or } x = 2. \end{aligned}$$

Thus the critical points are $x = -4$ and $x = 2$.

(b) In this case $f'(x) = -3x^2 + 18x - 24$. Thus we have to solve the equation

$$\begin{aligned} -3x^2 + 18x - 24 = 0 &\Leftrightarrow x^2 - 6x + 8 = 0 \\ &\Leftrightarrow (x-4)(x-2) = 0 \\ &\Leftrightarrow x = 2 \text{ or } x = 4. \end{aligned}$$

Thus the critical points are $x = 2$ and $x = 4$.

(c) In this case $f'(x) = -4e^{-4x} + 5$. Thus we have to solve the equation

$$\begin{aligned} -4e^{-4x} + 5 = 0 &\Leftrightarrow -4e^{-4x} = -5 \\ &\Leftrightarrow e^{-4x} = \frac{5}{4} \\ &\Leftrightarrow \ln\left(e^{-4x}\right) = \ln\left(\frac{5}{4}\right) \\ &\Leftrightarrow -4x = \ln\left(\frac{5}{4}\right) \\ &\Leftrightarrow x = -\frac{1}{4}\ln\left(\frac{5}{4}\right). \end{aligned}$$

So there is one critical point of f, that is $x = -\frac{1}{4}\ln\left(\frac{5}{4}\right)$.

(d) In this case $f'(x) = 2\cos(2x)$. Thus we have to solve the equation

$$\begin{aligned} 2\cos(2x) = 0 &\Leftrightarrow \cos(2x) = 0 \\ &\Leftrightarrow 2x = \frac{\pi}{2} + k\pi \quad \text{where } k \in \mathbb{Z} \\ &\Leftrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{where } k \in \mathbb{Z} \end{aligned}$$

That is, the critical points are $x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{where } k \in \mathbb{Z}$.

3. In these questions we will find where the global maxima and minima of each of the following functions occur by evaluating the functions at the endpoints of the domain and at any critical points that lie in the domain.

   Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto -x^3 + 9x^2 - 24x + 10 \end{aligned}$$

   have already been found in Question 2(b): they are $x = 2$ and $x = 4$.

   (a) In this case we evaluate $f(x)$ at $x = 0, 2, 4, 6$. $f(0) = 10$, $f(2) = -10$, $f(4) = -6$ and $f(6) = -26$. Hence the global maximum of f is 10 attained at $x = 0$ and the global minimum of f is -26 attained at $x = 6$.

   (b) In this case we evaluate $f(x)$ at $x = 1, 2, 4, 6$. $f(1) = -6$, $f(2) = -10$, $f(4) = -6$ and $f(6) = -26$. Hence the global maximum of f is -6 attained at $x = 1$ and $x = 4$, and the global minimum of f is -26 attained at $x = 6$.

(c) In this case we evaluate $f(x)$ at $x = 0, 2, 4, 5$. $f(0) = 10$, $f(2) = -10$, $f(4) = -6$ and $f(5) = -10$. Hence the global maximum of f is 10 attained at $x = 0$ and the global minimum of f is -10 attained at $x = 2$ and $x = 5$.

(d) In this case we evaluate $f(x)$ at $x = 2, 4$. $f(2) = -10$ and $f(4) = -6$. Hence the global maximum of f is -6 attained at $x = 4$ and the global minimum of f is -10 attained at $x = 2$.

(e) In this case we evaluate $f(x)$ at $x = 2, 3$. $f(2) = -10$ and $f(3) = -8$. Hence the global maximum of f is -8 attained at $x = 3$ and the global minimum of f is -10 attained at $x = 2$.

(f) In this case we evaluate $f(x)$ at $x = 3, 4$. $f(3) = -8$ and $f(4) = -6$. Hence the global maximum of f is -6 attained at $x = 4$ and the global minimum of f is -8 attained at $x = 3$.

Note that the critical point of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{-4x} + 5x \end{aligned}$$

has already been found in Question 2(c): it is $x = -\frac{1}{4}\ln\left(\frac{5}{4}\right)$.

(g) In this case we evaluate $f(x)$ at $x = -2, -1$.
$f(-2) = e^8 - 10 \simeq 2971$ and $f(-1) = e^4 - 5 \simeq 50$.
Hence the global maximum of f is $e^8 - 10$ attained at $x = -2$ and the global minimum of f is $e^4 - 5$ attained at $x = -1$.

(h) In this case we evaluate $f(x)$ at $x = -1, -\frac{1}{4}\ln\left(\frac{5}{4}\right), 0$.
$f(-1) = e^4 - 5 \simeq 50$,
$f\left(-\frac{1}{4}\ln\left(\frac{5}{4}\right)\right) = \exp\left(\ln\left(\frac{5}{4}\right)\right) - \frac{5}{4}\ln\left(\frac{5}{4}\right) = \frac{5}{4} - \frac{5}{4}\ln\left(\frac{5}{4}\right) \simeq 0.97$ and
$f(0) = e^0 - 0 = 1$.
Hence the global maximum of f is $e^4 - 5$ attained at $x = -1$ and the global minimum of f is $\frac{5}{4} - \frac{5}{4}\ln\left(\frac{5}{4}\right)$ attained at $x = -\frac{1}{4}\ln\left(\frac{5}{4}\right)$.

(i) In this case we evaluate $f(x)$ at $x = 0, 1$.
$f(0) = e^0 + 0 = 1$ and $f(1) = e^{-4} + 5 \simeq 5$.
Hence the global maximum of f is $e^{-4} + 5$ attained at $x = 1$ and the global minimum of f is 1 attained at $x = 0$.

4. Note that we have found all the derivatives and critical points of these functions in Question 2.

(a) Since $f'(x) = 3x^2 + 6x - 24$, $f''(x) = 6x + 6$. We now evaluate $f''(x)$ at each of the critical points.
$f''(-4) = 6(-4) + 6 = -18 < 0$, so the critical point at $x = -4$ is a local maximum.
$f''(2) = 6(2) + 6 = 18 > 0$, so the critical point at $x = 2$ is a local minimum.

(b) Since $f'(x) = -3x^2 + 18x - 24$, $f''(x) = -6x + 18$.
Note it is absolutely essential to use the correct $f'(x)$ here and **NOT** x^2-x-6, which we only used in the process of solving $f'(x) = 0$.
We now evaluate $f''(x)$ at each of the critical points.
$f''(2) = -6(2) + 18 = 6 > 0$, so the critical point at $x = 2$ is a local minimum.
$f''(4) = -6(4) + 18 = -6 < 0$, so the critical point at $x = 4$ is a local maximum.

(c) Since $f'(x) = -4e^{-4x} + 5$, $f''(x) = 16e^{-4x}$.
We now evaluate $f''(x)$ at the critical point $x = -\frac{1}{4}\ln\left(\frac{5}{4}\right)$.
$f''\left(-\frac{1}{4}\ln\left(\frac{5}{4}\right)\right) = 16\exp\left(\ln\left(\frac{5}{4}\right)\right) = 16\left(\frac{5}{4}\right) = 20$.
Since $f''\left(-\frac{1}{4}\ln\left(\frac{5}{4}\right)\right) > 0$, f has a local minimum at $x = -\frac{1}{4}\ln\left(\frac{5}{4}\right)$.

(d) To solve this problem we note that it appears (from looking at the graph of $f(x) = \sin(2x)$) that the maxima occur at $x = \frac{\pi}{4} + k\pi$ and the minima occur at $x = \frac{3\pi}{4} + k\pi$, where $k \in \mathbb{Z}$. We will now prove this using the second derivative test.
Since $f'(x) = 2\cos(2x)$, $f''(x) = -4\sin(2x)$.
We now evaluate $f''(x)$ at each of the critical points.

$$\begin{aligned} f''\left(\frac{\pi}{4} + k\pi\right) &= -4\sin\left(2\left(\frac{\pi}{4} + k\pi\right)\right) \\ &= -4\sin\left(\frac{\pi}{2} + 2k\pi\right) \\ &= -4(1) \\ &= -4 \\ &< 0, \end{aligned}$$

so the critical points at $x = \frac{\pi}{4} + k\pi$ are local maxima.

$$\begin{aligned} f''\left(\frac{3\pi}{4} + k\pi\right) &= -4\sin\left(2\left(\frac{3\pi}{4} + k\pi\right)\right) \\ &= -4\sin\left(\frac{3\pi}{2} + 2k\pi\right) \\ &= -4(-1) \\ &= 4 \\ &> 0, \end{aligned}$$

so the critical points at $x = \frac{3\pi}{4} + k\pi$ are local minima.

5. In this question we will use the iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

(a) In this case $f'(x) = 3x^2 - 18x + 12$, so that $x_{n+1} = x_n - \dfrac{x^3 - 9x^2 + 12x + 10}{3x^2 - 18x + 12}$. Since $x_0 = 2$,

$$x_1 = x_0 - \frac{x_0^3 - 9x_0^2 + 12x_0 + 10}{3x_0^2 - 18x_0 + 12} = 2 - \frac{2^3 - 9(2^2) + 12(2) + 10}{3(2^2) - 18(2) + 12} = \frac{5}{2}.$$

Then

$$x_2 = x_1 - \frac{x_1^3 - 9x_1^2 + 12x_1 + 10}{3x_1^2 - 18x_1 + 12} = \frac{5}{2} - \frac{\left(\frac{5}{2}\right)^3 - 9\left(\frac{5}{2}\right)^2 + 12\left(\frac{5}{2}\right) + 10}{3\left(\frac{5}{2}\right)^2 - 18\left(\frac{5}{2}\right) + 12} = \frac{140}{57}.$$

(b) In this case $f'(x) = -2\sin(2x) - 1$, so that $x_{n+1} = x_n - \dfrac{\cos(2x_n) - x_n}{-2\sin(2x_n) - 1}$. Since $x_0 = 1$,

$$x_1 = x_0 - \frac{\cos(2x_0) - x_0}{-2\sin(2x_0) - 1} = 1 - \frac{\cos(2) - 1}{-2\sin(2) - 1} \simeq 0.49756992.$$

Note that we have to use radians in this question - anytime calculus is involved, using degrees will lead to a mistake. Also note that we should keep x_1 to full calcultor accuracy. Then

$$\begin{aligned} x_2 &= x_1 - \frac{\cos(2x_1) - x_1}{-2\sin(2x_1) - 1} \\ &\simeq 0.49756992 - \frac{\cos(0.99513984) - 0.49756992}{-2\sin(0.99513984) - 1} \\ &\simeq 0.515053653. \end{aligned}$$