Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Assignment 4 Solutions

1. In all these solutions c will represent an arbitrary constant.

   (a) Here we use integration by substitution.
   Let $u = 2x + 1$, so that $\dfrac{du}{dx} = 2$.
   Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{2}$.
   Hence

$$\begin{aligned}\int (2x+1)^{12}\,dx &= \int u^{12}\cdot\frac{du}{2}\\ &= \frac{1}{2}\int u^{12}\,du\\ &= \frac{1}{26}u^{13}+c\\ &= \frac{1}{26}(2x+1)^{13}+c.\end{aligned}$$

   (b) Again we use integration by substitution.
   Let $u = x^5 - x^4 + x^3 - 1$, so that $\dfrac{du}{dx} = 5x^4 - 4x^3 + 3x^2$.
   Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{5x^4 - 4x^3 + 3x^2}$.
   Hence

$$\begin{aligned}\int \frac{10x^4 - 8x^3 + 6x^2}{x^5 - x^4 + x^3 - 1}\,dx &= \int \frac{10x^4 - 8x^3 + 6x^2}{u}\cdot\frac{du}{5x^4 - 4x^3 + 3x^2}\\ &= 2\int \frac{1}{u}\,du\\ &= 2\ln(u)+c\\ &= 2\ln(x^5 - x^4 + x^3 - 1)+c.\end{aligned}$$

   (c) Again we use integration by substitution.
   Let $u = x^3 + 2$, so that $\dfrac{du}{dx} = 3x^2$.
   Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{3x^2}$.

Hence

$$\begin{aligned}\int x^2\cos\left(x^3+2\right)\,dx &= \int x^2\cos(u)\cdot\frac{du}{3x^2}\\ &= \frac{1}{3}\int \cos(u)\,du\\ &= \frac{1}{3}\sin(u)+c\\ &= \frac{1}{3}\sin(x^3+2)+c.\end{aligned}$$

(d) Here we use integration by parts.
Let $f(x)=2x$ and $g'(x)=e^{3x}$, so that $f'(x)=2$ and $g(x)=\frac{1}{3}e^{3x}$.
Hence, using the integration by parts formula,

$$\int 2xe^{3x}\,dx = 2x\cdot\frac{1}{3}e^{3x}-\int\frac{2}{3}e^{3x}\,dx=\frac{2}{3}xe^{3x}-\frac{2}{9}e^{3x}+c.$$

(e) Again we use integration by parts.
Let $f(x)=\ln(x)$ and $g'(x)=x^2$, so that $f'(x)=\frac{1}{x}$ and $g(x)=\frac{1}{3}x^3$.
Hence, using the integration by parts formula,

$$\begin{aligned}\int x^2\ln(x)\,dx &= \ln(x)\cdot\frac{1}{3}x^3-\int\frac{1}{x}\cdot\frac{1}{3}x^3\,dx\\ &= \frac{1}{3}x^3\ln(x)-\int\frac{1}{3}x^2\,dx\\ &= \frac{1}{3}x^3\ln(x)-\frac{1}{9}x^3+c.\end{aligned}$$

(f) Here we use partial fractions.
Since $x^2+x-6=(x+3)(x-2)$, we let

$$\frac{-5}{x^2+x-6}=\frac{A}{x+3}+\frac{B}{x-2}, \tag{1}$$

where A and B are constants we have to find. Multiplying both sides of (1) by x^2+x-6 we obtain

$$-5=A(x-2)+B(x+3). \tag{2}$$

If we let $x=2$ in (2), we obtain $-5=5B$, so that $B=-1$.
If we let $x=-3$ in (2), we obtain $-5=-5A$, so that $A=1$.
Hence

$$\begin{aligned}\int\frac{-5}{x^2+x-6}\,dx &= \int\frac{1}{x+3}+\frac{-1}{x-2}\,dx\\ &= \int\frac{1}{x+3}\,dx-\int\frac{1}{x-2}\,dx\\ &= \ln(x+3)-\ln(x-2)+c.\end{aligned}$$

(g) Again we use partial fractions.
Since $x^2 + x + 1$ can't be factored, we let

$$\frac{3x^2}{(x^2 + x + 1)(x - 1)} = \frac{Ax + B}{x^2 + x + 1} + \frac{C}{x - 1}, \tag{3}$$

where A, B and C are constants we have to find. Multiplying both sides of (1) by $(x^2 + x + 1)(x - 1)$ we obtain

$$3x^2 = (Ax + B)(x - 1) + C(x^2 + x + 1). \tag{4}$$

Multiplying this out and collecting terms we get

$$3x^2 = (A + C)x^2 + (-A + B + C)x + (-B + C). \tag{5}$$

Comparing coefficients of powers of x, we get the simultaneous equations $A+C = 3$, $-A+B+C = 0$ and $-B+C = 0$. The last of these gives $B = C$ and the first gives $A = 3-C$. Substituting for A and B in $-A+B+C = 0$, we get $-(3 - C) + C + C = 0$, so that $C = 1$. Then $B = C = 1$ and $A = 3 - C = 2$. Hence

$$\begin{aligned}\int \frac{3x^2}{(x^2 + x + 1)(x - 1)}\,dx &= \int \frac{2x + 1}{x^2 + x + 1} + \frac{1}{x - 1}\,dx \\ &= \int \frac{2x + 1}{x^2 + x + 1}\,dx + \int \frac{1}{x - 1}\,dx \\ &= \ln(x^2 + x + 1) + \ln(x - 1) + c.\end{aligned}$$

I found $\int \frac{2x + 1}{x^2 + x + 1}\,dx$ 'by inspection', but you can use the substitution $u = x^2 + x + 1$ if you can't spot it.

2. (a) Here we use integration by substitution.
Let $u = 3x^4 - 3$, so that so that $\frac{du}{dx} = 12x^3$.
Then $dx = \frac{du}{du/dx} = \frac{du}{12x^3}$.
Also, when $x = 1$, $u = 0$ and when $x = -1$, $u = 0$.
Hence

$$\begin{aligned}\int_{-1}^{1} x^3(3x^4 - 3)^8\,dx &= \int_0^0 x^3u^8 \cdot \frac{du}{12x^3} \\ &= \int_0^0 \frac{1}{12}u^8\,du \\ &= 0.\end{aligned}$$

Note that if you have an integral where the upper lower limits are the same then the integral is zero.

(b) Again we will use integration by substitution.
Let $u = 2\sin(x) - 3\cos(x)$, so that so that $\frac{du}{dx} = 2\cos(x) + 3\sin(x)$.

Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{2\cos(x) + 3\sin(x)}$.

Also, when $x = \pi$, $u = 3$ and when $x = \dfrac{\pi}{2}$, $u = 2$.

Hence

$$\begin{aligned}
\int_{\frac{\pi}{2}}^{\pi} \frac{4\cos(x) + 6\sin(x)}{2\sin(x) - 3\cos(x)}\,dx &= \int_2^3 \frac{4\cos(x) + 6\sin(x)}{u} \cdot \frac{du}{2\cos(x) + 3\sin(x)} \\
&= \int_2^3 \frac{2}{u}\,du \\
&= [2\ln(u)]_2^3 \\
&= 2\ln(3) - 2\ln(2) \\
&= 2(\ln(3) - \ln(2)).
\end{aligned}$$

(c) Again we will use integration by substitution.

Let $u = x^3 - x^2 + 1$, so that so that $\dfrac{du}{dx} = 3x^2 - 2x$.

Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{3x^2 - 2x}$.

Also, when $x = 1$, $u = 1$ and when $x = 0$, $u = 1$.

Hence

$$\begin{aligned}
\int_1^1 (3x^2 - 2x)(x^3 - x^2 + 1)^{\frac{5}{2}}\,dx &= \int_1^1 (3x^2 - 2x)u^{\frac{5}{2}} \cdot \frac{du}{3x^2 - 2x} \\
&= \int_1^1 u^{\frac{5}{2}}\,du \\
&= 0.
\end{aligned}$$

Again, the upper and lower limits are equal so the integral is zero.

(d) This is another integration by substitution.

Let $u = 2\cos(x)$, so that so that $\dfrac{du}{dx} = -2\sin(x)$.

Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{-2\sin(x)}$.

Also, when $x = \dfrac{\pi}{2}$, $u = 0$ and when $x = -\dfrac{\pi}{2}$, $u = 0$.

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(x)e^{2\cos(x)}\,dx = \int_0^0 \sin(x)e^u \cdot \frac{du}{-2\sin(x)} = \int_0^0 -\frac{1}{2}e^u\,du = 0.$$

(e) Here we have to use integration by parts.

Let $f(x) = x$ and $g'(x) = \cos(x)$, so that $f'(x) = 1$ and $g(x) = \sin(x)$.

Hence, using the integration by parts formula,

$$\begin{aligned}
\int_{-\pi}^0 x\cos(x)\,dx &= [x\sin(x)]_{-\pi}^0 - \int_{-\pi}^0 \sin(x)\,dx \\
&= 0 - (-\pi(0)) - [-\cos(x)]_{-\pi}^0 \\
&= [\cos(x)]_{-\pi}^0 \\
&= 1 - (-1) \\
&= 2.
\end{aligned}$$

(f) This is another integration by parts.
Let $f(x) = \ln(x)$ and $g'(x) = x^3 + 1$, so that $f'(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{4}x^4 + x$.
Hence, using the integration by parts formula,

$$\begin{aligned}
\int_1^2 (x^3+1)\ln(x)\,dx &= \left[\ln(x)\cdot\left(\frac{1}{4}x^4 + x\right)\right]_1^2 - \int_1^2 \frac{1}{x}\cdot\left(\frac{1}{4}x^4 + x\right)\,dx \\
&= \ln(2)\left(\frac{1}{4}2^4 + 2\right) - \ln(1)\left(\frac{1}{4}1^4 + 1\right) - \int_1^2 \left(\frac{1}{4}x^3 + 1\right)\,dx \\
&= 6\ln(2) - \left[\frac{1}{16}x^4 + x\right]_1^2 \\
&= 6\ln(2) - \left[\frac{1}{16}2^4 + 2 - \left(\frac{1}{16}1^4 + 1\right)\right] \\
&= 6\ln(2) - \frac{31}{16}.
\end{aligned}$$

(g) Here we use partial fractions.
Since $x^2 - 4x + 3 = (x-3)(x-1)$, we let

$$\frac{3x-7}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1}, \tag{6}$$

where A and B are constants we have to find. Multiplying both sides of (6) by $x^2 - 4x + 3$ we obtain

$$3x - 7 = A(x-1) + B(x-3). \tag{7}$$

If we let $x = 1$ in (7), we obtain $-4 = -2B$, so that $B = 2$.
If we let $x = 3$ in (7), we obtain $2 = 2A$, so that $A = 1$.
Hence

$$\begin{aligned}
\int_4^5 \frac{3x-7}{x^2-4x+3}\,dx &= \int_4^5 \frac{1}{x-3} + \frac{2}{x-1}\,dx \\
&= [\ln(x-3) + 2\ln(x-1)]_4^5 \\
&= \ln(2) + 2\ln(4) - (\ln(1) + 2\ln(3)) \\
&= \ln(2) + 2(\ln(4) - \ln(3)).
\end{aligned}$$

(h) Again we use partial fractions.
This time we let

$$\frac{2x^2-3x+5}{(x+1)^2(x-4)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x-4}, \tag{8}$$

where A, B and C are constants we have to find. Multiplying both sides of (8) by $(x+1)^2(x-4)$ we obtain

$$2x^2 - 3x + 5 = A(x-4) + B(x+1)(x-4) + C(x+1)^2. \tag{9}$$

Multiplying this out and collecting terms we get

$$2x^2 - 3x + 5 = (B+C)x^2 + (A - 3B + 2C)x + (-4A - 4B + C). \tag{10}$$

Comparing coefficients of powers of x in (10), we get the simultaneous equations $B + C = 2$, $A - 3B + 2C = -3$ and $-4A - 4B + C = 5$.
From the first equation we get $C = 2 - B$ and if we substitute this into the other two equations, we get $A - 3B + 2(2 - B) = -3$ and $-4A - 4B + 2 - B = 5$.
That is $A - 5B = -7$ and $-4A - 5B = 3$.
Subtracting the second of these equations from the first we obtain $5A = -10$, so that $A = -2$.
Then, using $A - 5B = -7$, we get $5B = A + 7 = 5$, so that $B = 1$.
Then finally $C = 2 - B = 1$.

$$\begin{aligned}
\int_5^6 \frac{2x^2 - 3x + 5}{(x+1)^2(x-4)}\,dx &= \int_5^6 \frac{-2}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x-4}\,dx \\
&= \left[\frac{2}{x+1} + \ln(x+1) + \ln(x-4)\right]_5^6 \\
&= \frac{2}{7} + \ln(7) + \ln(2) - \left(\frac{2}{6} + \ln(6) + \ln(1)\right) \\
&= -\frac{1}{21} + \ln(7) + \ln(2) - \ln(6).
\end{aligned}$$

Note that I integrated $\dfrac{-2}{(x+1)^2}$ 'by inspection' but you can use the substitution $u = x + 1$ if you can't spot it.

3. (a) The graph of $f(x) = x^5$ lies below the x-axis between $x = -1$ and $x = 0$ and above the x-axis between $x = 0$ and $x = 1$. Thus the required area is

$$\begin{aligned}
-\int_{-1}^0 x^5\,dx + \int_0^1 x^5\,dx &= -\left[\frac{1}{6}x^6\right]_{-1}^0 + \left[\frac{1}{6}x^6\right]_0^1 \\
&= -\left[0 - \frac{1}{6}(-1)^6\right] + \left[\frac{1}{6}1^6 - 0\right] \\
&= \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3}.
\end{aligned}$$

(b) The graph of $f(x) = \sin(3x)$ lies above the x-axis between the points $x = 0$ and $x = \frac{\pi}{3}$. Thus the required area is

$$\int_0^{\frac{\pi}{3}} \sin(3x)\,dx = \left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{3}} = -\frac{1}{3}(-1) - \left(-\frac{1}{3}(1)\right) = \frac{2}{3}.$$

(c) The graph of $f(x) = e^{\frac{x}{2}}$ lies above the x-axis between the points $x = 0$ and $x = 1$. Thus the required area is

$$\int_0^1 e^{\frac{x}{2}}\,dx = \left[2e^{\frac{x}{2}}\right]_0^1 = 2e^{\frac{1}{2}} - 2e^0 = 2e^{\frac{1}{2}} - 2$$

(d) Since $f(0) = 6$ and the graph of $f(x) = x^3 - 2x^2 - 5x + 6$ only crosses the x-axis at $x = 1$ between the points $x = 0$ and $x = 2$, it follows that the graph of $f(x) = x^3 - 2x^2 - 5x + 6$ lies above the x-axis between the points $x = 0$ and $x = 1$ and below the x-axis between the points $x = 1$ and $x = 2$. Hence the required area is

$$\int_0^1 x^3 - 2x^2 - 5x + 6\,dx - \int_1^2 x^3 - 2x^2 - 5x + 6\,dx$$
$$= \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\right]_0^1 - \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x\right]_1^2$$
$$= \left[\left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right) - 0\right]$$
$$- \left[\left(\frac{1}{4}2^4 - \frac{2}{3}2^3 - \frac{5}{2}2^2 + 6(2)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6\right)\right]$$
$$= \frac{37}{12} - \left[\frac{2}{3} - \frac{37}{12}\right]$$
$$= \frac{11}{2}.$$

4. (a) Using the formula $V = \pi \int_a^b f(x)^2\,dx$, the volume is

$$V = \pi \int_0^1 x^2\,dx = \pi\left[\frac{1}{3}x^3\right]_0^1 = \pi\left(\frac{1}{3} - 0\right) = \frac{1}{3}\pi.$$

(b) The volume is

$$V = \pi \int_{-1}^1 1 - x^2\,dx$$
$$= \pi\left[x - \frac{1}{3}x^3\right]_{-1}^1$$
$$= \pi\left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{1}{3}(-1)^3\right)\right]$$
$$= \pi\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right]$$
$$= \frac{4}{3}\pi.$$

(c) The volume is

$$V = \pi \int_0^1 (e^x)^2\,dx = \pi \int_0^1 e^{2x}\,dx = \pi\left[\frac{1}{2}e^{2x}\right]_0^1 = \pi\left[\frac{1}{2}e^2 - \frac{1}{2}e^0\right] = \frac{e^2 - 1}{2}\pi.$$

(d) The volume is

$$
\begin{aligned}
\pi \int_0^{\frac{\pi}{2}} \left(\sqrt{\cos(x)}\right)^2 \, dx &= \pi \int_0^{\frac{\pi}{2}} \cos(x) \, dx \\
&= \pi \left[\sin(x)\right]_0^{\frac{\pi}{2}} \\
&= \pi[\sin\left(\frac{\pi}{2}\right) - (\sin(0))] \\
&= \pi(1-0) \\
&= \pi.
\end{aligned}
$$