Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Chapter 1 Solutions

1. (a) $5\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 15 & -5 \\ 0 & 10 \end{pmatrix}$

   (b) $\frac{1}{5}\begin{pmatrix} 2 & -5 \\ 10 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -1 \\ 2 & \frac{3}{5} \end{pmatrix}$

   (c) $-\frac{2}{3}\begin{pmatrix} 3 & -5 \\ 4 & -9 \end{pmatrix} = \begin{pmatrix} -2 & \frac{10}{3} \\ -\frac{8}{3} & 6 \end{pmatrix}$

2. (a) (i) $a_{23} = 1$

   (ii) $d_{21} = -6$

   (iii) $e_{12} = 0$

   (iv) f_{33} has no meaning since F only has two rows.

   (b) A is a 2×3 matrix, B is a 2×2 matrix, C is a 1×3 matrix, D is a 3×2 matrix, E is a 2×2 matrix and F is a 2×3 matrix.

   (c) (i) The combinations that can be formed are $A + F$, $B + B$, $B + E$ and $F - A$.

   (ii)

$$A + F = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \end{pmatrix} + \begin{pmatrix} -6 & -5 & 3 \\ -1 & 5 & -3 \end{pmatrix} = \begin{pmatrix} -4 & -6 & 8 \\ -1 & 8 & -2 \end{pmatrix}$$

$$B + B = \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -4 & 8 \\ 10 & -6 \end{pmatrix}$$

$$B + E = \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 3 & -2 \end{pmatrix}$$

$$F - A = \begin{pmatrix} -6 & -5 & 3 \\ -1 & 5 & -3 \end{pmatrix} - \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -8 & -4 & -2 \\ -1 & 2 & -4 \end{pmatrix}$$

   (d) (i) All are defined except AB.

   (ii) BF is a 2×3 matrix, BE is a 2×2 matrix, EB is a 2×2 matrix, FD is a 2×2 matrix, DA is a 3×3 matrix, B^3 is a 2×2 matrix, $(CD)E$ is a 1×2 matrix and $C(DE)$ is a 1×2 matrix.

(iii)

$$BF = \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -6 & -5 & 3 \\ -1 & 5 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 30 & -18 \\ -27 & -40 & 24 \end{pmatrix}$$

$$BE = \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -20 & 4 \\ 36 & -3 \end{pmatrix}$$

$$EB = \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} -12 & 24 \\ 9 & -11 \end{pmatrix}$$

$$FD = \begin{pmatrix} -6 & -5 & 3 \\ -1 & 5 & -3 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -6 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 15 & -9 \\ -43 & -5 \end{pmatrix}$$

$$DA = \begin{pmatrix} 4 & 2 \\ -6 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 5 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 2 & 22 \\ -12 & 6 & -30 \\ 6 & 0 & 16 \end{pmatrix}$$

$$\begin{aligned} B^3 &= \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 4 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} 24 & -20 \\ -25 & 29 \end{pmatrix} \\ &= \begin{pmatrix} -148 & 156 \\ 195 & -187 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} (CD)E = C(DE) &= \begin{pmatrix} 2 & -4 & 7 \end{pmatrix} \left[\begin{pmatrix} 4 & 2 \\ -6 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ -2 & 1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 2 & -4 & 7 \end{pmatrix} \begin{pmatrix} 20 & 2 \\ -36 & 0 \\ 16 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 296 & 11 \end{pmatrix} \end{aligned}$$

3. (a) A is a 2×2 matrix, B is a 2×3 matrix, C is a 2×3 matrix and D is a 3×2 matrix.

   (b) (i) They can all be formed except $A^T + B$ and $3C^T + A$.

(ii)

$$2B^T + 3C^T = 2\begin{pmatrix} 4 & 6 & 5 \\ 2 & 1 & 3 \end{pmatrix}^T + 3\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 5 \end{pmatrix}^T$$
$$= 2\begin{pmatrix} 4 & 2 \\ 6 & 1 \\ 5 & 3 \end{pmatrix} + 3\begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 2 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & 4 \\ 12 & 2 \\ 10 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ 12 & 9 \\ 6 & 15 \end{pmatrix}$$
$$= \begin{pmatrix} 11 & 10 \\ 24 & 11 \\ 16 & 21 \end{pmatrix}$$

$$2C^T - B^T = 2\begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 5 \end{pmatrix}^T - \begin{pmatrix} 4 & 6 & 5 \\ 2 & 1 & 3 \end{pmatrix}^T$$
$$= 2\begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 2 & 5 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 6 & 1 \\ 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 4 \\ 8 & 6 \\ 4 & 10 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 6 & 1 \\ 5 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 2 \\ 2 & 5 \\ -1 & 7 \end{pmatrix}$$

$$2D + 3D = 5D = 5\begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 20 & 0 \\ 0 & 15 \end{pmatrix}$$

$$\begin{aligned}
A - 2A^T + A &= 2A - 2A^T \\
&= 2(A - A^T) \\
&= 2\left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}^T\right] \\
&= 2\left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}\right] \\
&= 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}
\end{aligned}$$

(c) (i) They are all defined except CA^T, $(D^2)^T$ and $(D^T)^2$.

(ii) AB is a 2×3 matrix, $C^T D^T$ is a 3×3 matrix, AD^T is a 2×3 matrix, DB is a 3×3 matrix, $(A^2)^T$ is a 2×2 matrix and $(A^T)^2$ is a 2×2 matrix.

(iii)

$$AB = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 4 & 6 & 5 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 12 & 10 \\ 10 & 9 & 14 \end{pmatrix}$$

$$\begin{aligned}
C^T D^T &= \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 5 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 0 & 3 \end{pmatrix}^T \\
&= \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 4 & 6 \\ 10 & 16 & 9 \\ 12 & 8 & 15 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
AD^T &= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 0 & 3 \end{pmatrix}^T \\
&= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 8 & 0 \\ 7 & 4 & 9 \end{pmatrix}
\end{aligned}$$

$$DB = \begin{pmatrix} 1 & 2 \\ 4 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 6 & 5 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 11 \\ 16 & 24 & 20 \\ 6 & 3 & 9 \end{pmatrix}$$

$$(A^2)^T = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \right]^T = \begin{pmatrix} 4 & 0 \\ 5 & 9 \end{pmatrix}^T = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$$

$$(A^T)^2 = A^T A^T = (AA)^T = (A^2)^T = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} \text{ from above,}$$

$$\begin{aligned} \text{or } (A^T)^2 &= \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}^T \right]^2 \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}^2 \\ &= \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} \end{aligned}$$

4. (a) $\det(A) = (1)(4) - (3)(2) = -2 \neq 0$, so A has an inverse.
$\det(B) = (1)(1) - (0)(0) = 1 \neq 0$, so B has an inverse.
$\det(C) = (-1)(4) - (2)(-3) = 2 \neq 0$, so C has an inverse.
$\det(D) = (2)(3) - (2)(3) = 0$, so D does not have an inverse.
$\det(E) = (1)(10) - (5)(-2) = 20 \neq 0$, so E has an inverse.
$\det(F) = (2)(0) - (4)(-3) = 12 \neq 0$, so F has an inverse.
$\det(G) = (0)(-5) - (0)(2) = 0$, so G does not have an inverse.
$\det(H) = (-2)(-6) - (-3)(-1) = 9 \neq 0$, so H has an inverse.

(b)

$$A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

$$B^{-1} = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that in this case we have $B^{-1} = B$.

$$C^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$E^{-1} = \frac{1}{20} \begin{pmatrix} 10 & -5 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{10} & \frac{1}{20} \end{pmatrix}$$

$$F^{-1} = \frac{1}{12}\begin{pmatrix} 0 & -4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{pmatrix}$$

$$H^{-1} = \frac{1}{9}\begin{pmatrix} -6 & 3 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{9} & -\frac{2}{9} \end{pmatrix}$$

5. (a) We will row reduce the augmented matrix $\begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 5 \\ 1 & -1 & 0 & 5 \end{pmatrix}$.

$$\begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 - R1 \end{array} \quad \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 2 \end{pmatrix}$$

$$R2 \leftrightarrow R3 \quad \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & -2 & -2 & 2 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\begin{array}{l} R2 \to -\frac{1}{2}R2 \\ R3 \to -R3 \end{array} \quad \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{l} R1 \to R1 - 2R3 \\ R2 \to R2 - R3 \end{array} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$R1 \to R1 - R2 \quad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence the solution is $x = 3,\ y = -2,\ z = 1$

(b) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & -2 \end{pmatrix}$.

$$R3 \to R3 - R1 \quad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -2 \end{pmatrix}$$

$$R3 \to R3 + R2 \quad \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \to R1 - 2R2 \quad \begin{pmatrix} 1 & 0 & 2 & -4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is no leading 1 in the z column, z is a free variable. Thus we can let $z = t$, $t \in \mathbb{R}$. From the top row we then have $x + 2z = -4$, so $x = -4 - 2z = -4 - 2t$. Finally from the second row we have $y - z = 2$, so $y = 2 + z = 2 + t$. Thus the solution is $x = -4 - 2t$, $y = 2 + t$, $z = t$ for any real number t.

(c) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 5 & 10 \end{pmatrix}$.

$$R3 \to R3 - R1 \quad \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{pmatrix}$$

$$R3 \to R3 - R2 \quad \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Now, the system of equations corresponding to this augmented matrix has as its third equation $0x + 0y + 0z = 3$. For all values of x, y and z, this reduces to $0 = 3$ which is clearly impossible. Thus the system of equations has no solutions.

(d) Again we will row reduce the augmented matrix $\begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 8 \\ -1 & 3 & 2 & -8 \end{pmatrix}$.

$$R1 \leftrightarrow R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 2 & -1 & 1 & 1 \\ -1 & 3 & 2 & -8 \end{pmatrix}$$

$$\begin{matrix} R2 \to R2 - 2R1 \\ R3 \to R3 + R1 \end{matrix} \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & -3 & 3 & -15 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$R2 \to -\tfrac{1}{3}R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$R3 \to R3 - 4R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 5 & -20 \end{pmatrix}$$

$$R3 \to \tfrac{1}{5}R3 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

$$\begin{array}{lcl} R1 & \to & R1 + R3 \\ R2 & \to & R2 + R3 \end{array} \quad \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

$$R1 \to R1 - R2 \quad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Hence the solution is $x = 3,\ y = 1,\ z = -4$.

(e) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 1 & 1 & 6 \\ -1 & 1 & -3 & -2 \\ 2 & 1 & 3 & 6 \end{pmatrix}$.

$$\begin{array}{lcl} R2 & \to & R2 + R1 \\ R3 & \to & R3 - 2R1 \end{array} \quad \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -6 \end{pmatrix}$$

$$R2 \to \tfrac{1}{2}R2 \quad \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 1 & -6 \end{pmatrix}$$

$$R3 \to R3 + R2 \quad \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Now, the system of equations corresponding to this augmented matrix has as its third equation $0x + 0y + 0z = -4$. For all values of x, y and z, this reduces to $0 = -4$ which is clearly impossible. Thus the system of equations has no solutions.

(f) Again we will row reduce the augmented matrix $\begin{pmatrix} -1 & 2 & -4 & 6 \\ 1 & 3 & -1 & 4 \\ 1 & 2 & 0 & 2 \end{pmatrix}$.

$$R1 \to -R1 \quad \begin{pmatrix} 1 & -2 & 4 & -6 \\ 1 & 3 & -1 & 4 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

$$\begin{array}{lcl} R2 & \to & R2 - R1 \\ R3 & \to & R3 - R1 \end{array} \quad \begin{pmatrix} 1 & -2 & 4 & -6 \\ 0 & 5 & -5 & 10 \\ 0 & 4 & -4 & 8 \end{pmatrix}$$

$$R2 \to \tfrac{1}{5}R2 \quad \begin{pmatrix} 1 & -2 & 4 & -6 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & -4 & 8 \end{pmatrix}$$

$$\begin{array}{rcl} & & \begin{pmatrix} 1 & -2 & 4 & -6 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ R3 & \to & R3 - 4R2 \end{array}$$

$$\begin{array}{rcl} R1 & \to & R1 + 2R2 \end{array} \quad \begin{pmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is no leading 1 in the z column, z is a free variable. Thus we can let $z = t$, $t \in \mathbb{R}$. From the top row we then have $x + 2z = -2$, so $x = -2 - 2z = -2 - 2t$. Finally from the second row we have $y - z = 2$, so $y = 2 + z = 2 + t$. Thus the solution is $x = -2 - 2t$, $y = 2 + t$, $z = t$ for any real number t.

(g) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ -1 & 2 & -5 \end{pmatrix}$.

$$\begin{array}{rcl} R2 & \to & R2 - 2R1 \\ R3 & \to & R3 + R1 \end{array} \quad \begin{pmatrix} 1 & 4 & -7 \\ 0 & -9 & 18 \\ 0 & 6 & -12 \end{pmatrix}$$

$$R2 \to -\tfrac{1}{9}R2 \quad \begin{pmatrix} 1 & 4 & -7 \\ 0 & 1 & -2 \\ 0 & 6 & -12 \end{pmatrix}$$

$$R3 \to R3 - 6R2 \quad \begin{pmatrix} 1 & 4 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R1 \to R1 - 4R2 \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus the solution is $x = 1,\ y = -2$.

(h) Again we will row reduce the augmented matrix $\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 3 & -1 & 5 & 0 \end{pmatrix}$.

$$R3 \to R3 - 3R1 \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -4 & 8 & 0 \end{pmatrix}$$

$$R3 \to R3 + 4R2 \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \to R1 - R2 \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is no leading 1 in the z column, z is a free variable. Thus we can let $z = t$, $t \in \mathbb{R}$. From the top row we then have $x + z = 0$, so $x = -z = -t$. Finally from the second row we have $y - 2z = 0$, so $y = 2z = 2t$. Thus the solution is $x = -t$, $y = 2t$, $z = t$ for any real number t.

(i) Again we will row reduce the augmented matrix $\begin{pmatrix} 3 & -11 & -3 & 3 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{pmatrix}$.

$$R1 \to R1 - R2 \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{pmatrix}$$

$$\begin{aligned} R2 &\to R2 - 2R1 \\ R3 &\to R3 - 5R1 \\ R4 &\to R4 - 4R1 \end{aligned} \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 4 & 0 & -3 \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{pmatrix}$$

$$R2 \to \tfrac{1}{4}R2 \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{pmatrix}$$

$$\begin{aligned} R3 &\to R3 - 8R2 \\ R4 &\to R4 - 12R2 \end{aligned} \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$$R3 \to -R3 \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$$R4 \to R4 - 4R3 \quad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \rightarrow R1 + R3 \quad \begin{pmatrix} 1 & -5 & 0 & 4 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \rightarrow R1 + 5R2 \quad \begin{pmatrix} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence the solution is $x = \dfrac{1}{4},\ y = -\dfrac{3}{4},\ z = 2.$

(j) Again we will row reduce the augmented matrix $\begin{pmatrix} 4 & 2 & 1 & 0 \\ -2 & -1 & 5 & 11 \\ 6 & 3 & -1 & -5 \end{pmatrix}$.

$$R1 \rightarrow \tfrac{1}{4}R1 \quad \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ -2 & -1 & 5 & 11 \\ 6 & 3 & -1 & -5 \end{pmatrix}$$

$$\begin{array}{l} R2 \rightarrow R2 + 2R1 \\ R3 \rightarrow R3 - 6R1 \end{array} \quad \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{11}{2} & 11 \\ 0 & 0 & -\frac{5}{2} & -5 \end{pmatrix}$$

$$R2 \rightarrow \tfrac{2}{11}R2 \quad \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -\frac{5}{2} & -5 \end{pmatrix}$$

$$R3 \rightarrow R3 + \tfrac{5}{2}R2 \quad \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R1 \rightarrow R1 - \tfrac{1}{4}R2 \quad \begin{pmatrix} 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there is no leading 1 in the y column, y is a free variable. Thus we can let $y = t$, $t \in \mathbb{R}$. From the top row we then have $x + \frac{1}{2}y = -\frac{1}{2}$, so $x = -\frac{1}{2}y - \frac{1}{2} = -\frac{1}{2}t - \frac{1}{2}$. Finally from the seond row we have $z = 2$. Thus the solution is $x = -\frac{1}{2}t - \frac{1}{2}$, $y = t$, $z = 2$ for any real number t.

6. (a) We will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & 3 & -1 & 1 & 0 & 0 \\ -2 & -5 & 1 & 0 & 1 & 0 \\ 4 & 11 & -2 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{l} R2 \rightarrow R2 + 2R1 \\ R3 \rightarrow R3 - 4R1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & -4 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + R2 \quad \left(\begin{array}{ccc|ccc} 1 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

$$\begin{array}{l} R1 \to R1 + R3 \\ R2 \to R2 + R3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

$$R1 \to R1 - 3R2 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -5 & -2 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -5 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 1 \end{pmatrix}$.

(b) Again we will row reduce the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 3 & 1 & 7 & 0 & 1 & 0 \\ 2 & 4 & 8 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{l} R2 \to R2 - 3R1 \\ R3 \to R3 - 2R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & -8 & -8 & -3 & 1 & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$R2 \to -\tfrac{1}{8}R2 \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & -2 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + 2R2 \quad \left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -\frac{10}{8} & -\frac{1}{4} & 1 \end{array}\right)$$

At this stage we see that there is no leading 1 in the third column (and if we continue row reducing this will not change). Thus B has no inverse.

(c) Again we will row reduce the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & 0 & -4 & 0 & 1 & 0 \\ 3 & 2 & 10 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{l} R2 \to R2 + R1 \\ R3 \to R3 - 3R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -1 & 4 & -3 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + R2 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right)$$

$$R3 \to \tfrac{1}{2}R3 \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\begin{array}{l} R1 \to R1 - 2R3 \\ R2 \to R2 + 2R3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$R1 \to R1 - R2 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -2 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

Hence $C^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -3 & -2 \\ -1 & 2 & 1 \\ -1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

(d) Again we will row reduce the augmented matrix $\left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right)$.

$$R1 \leftrightarrow R2 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 1 & 0 \\ 2 & 4 & 6 & 1 & 0 & 0 \\ 5 & 10 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 - 5R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 0 \\ 0 & 0 & -15 & 0 & -5 & 1 \end{array}\right)$$

At this stage we see that there is no leading 1 in the second column (and if we continue row reducing this will not change). Thus D has no inverse.

(e) Again we will row reduce the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{l} R2 \to R2 + 2R1 \\ R3 \to R3 - R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 3 & 9 & 2 & 1 & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$R2 \to \tfrac{1}{3}R2 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & -3 & -1 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + R2 \quad \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 1 \end{array}\right)$$

At this stage we see that there is no leading 1 in the third column (and if we continue row reducing this will not change). Thus E has no inverse.

(f) Again we will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & -6 & 0 & 1 & 0 \\ -3 & -1 & 9 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{rcl} R2 & \to & R2 - 2R1 \\ R3 & \to & R3 + 3R1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right)$$

$$R2 \to -R2 \quad \left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 - 2R2 \quad \left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$\begin{array}{rcl} R1 & \to & R1 + 4R3 \\ R2 & \to & R2 + 2R3 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & -3 & 8 & 4 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

$$R1 \to R1 - R2 \quad \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 5 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)$$

Hence $F^{-1} = \begin{pmatrix} 1 & 1 & -4 \\ 2 & 1 & -6 \\ -3 & -1 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 5 & 2 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

(g) Again we will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & 4 & 1 & 1 & 0 & 0 \\ 1 & 6 & 3 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{rcl} R2 & \to & R2 - R1 \\ R3 & \to & R3 - 2R1 \end{array} \quad \left(\begin{array}{rrr|rrr} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & -5 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$R2 \to \tfrac{1}{2}R2 \quad \left(\begin{array}{rrr|rrr} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -5 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + 5R2 \quad \left(\begin{array}{rrr|rrr} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & -\frac{9}{2} & \frac{5}{2} & 1 \end{array}\right)$$

$$R3 \to \tfrac{1}{3}R3 \quad \left(\begin{array}{rrr|rrr} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{array}\right)$$

$$\begin{array}{rcl} R1 & \to & R1 - R3 \\ R2 & \to & R2 - R3 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{5}{2} & -\frac{5}{6} & -\frac{1}{3} \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{array}\right)$$

$$R1 \to R1 - 4R2 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{array}\right)$$

Hence $G^{-1} = \begin{pmatrix} 1 & 4 & 1 \\ 1 & 6 & 3 \\ 2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} & 1 \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{3}{2} & \frac{5}{6} & \frac{1}{3} \end{pmatrix}$

(h) Again we will row reduce the augmented matrix $\left(\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 1 & 6 & -3 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$.

$$\begin{array}{rcl} R2 & \to & R2 - R1 \\ R3 & \to & R3 + 2R1 \end{array} \quad \left(\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 10 & -4 & -1 & 1 & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right)$$

$$R2 \to \frac{1}{10}R2 \quad \left(\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & -5 & 2 & 2 & 0 & 1 \end{array}\right)$$

$$R3 \to R3 + 5R2 \quad \left(\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{5} & -\frac{1}{10} & \frac{1}{10} & 0 \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 \end{array}\right)$$

At this stage we see that there is no leading 1 in the third column (and if we continue row reducing this will not change). Thus H has no inverse.

7. (a)

$$\begin{aligned} \det(A) =& \det\begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 4 & 11 & -2 \end{pmatrix} \\ =& 1\begin{vmatrix} -5 & 1 \\ 11 & -2 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} + (-1)\begin{vmatrix} -2 & -5 \\ 4 & 11 \end{vmatrix} \\ =& 1(-5 \times (-2) - 1 \times 11) - 3(-2 \times (-2) - 1 \times 4) \\ & - 1(-2 \times 11 - (-5) \times 4) \\ =& 1(-1) - 3(0) - 1(-2) \\ =& -1 - 0 + 2 \\ =& 1. \end{aligned}$$

Since $\det(A)$ is non-zero, we have confirmed that A is invertible.

(b)

$$
\begin{aligned}
\det(B) =& \det\begin{pmatrix} 1 & 3 & 5 \\ 3 & 1 & 7 \\ 2 & 4 & 8 \end{pmatrix} \\
=& 1\begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} - 3\begin{vmatrix} 3 & 7 \\ 2 & 8 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} \\
=& 1(1 \times 8 - 7 \times 4) - 3(3 \times 8 - 7 \times 2) \\
& 5(3 \times 4 - 1 \times 2) \\
=& 1(-20) - 3(10) + 5(10) \\
=& -20 - 30 + 50 \\
=& 0.
\end{aligned}
$$

Since $\det(B) = 0$, we have confirmed that B is not invertible.

(c)

$$
\begin{aligned}
\det(C) =& \det\begin{pmatrix} 1 & 1 & 2 \\ -1 & 0 & -4 \\ 3 & 2 & 10 \end{pmatrix} \\
=& 1\begin{vmatrix} 0 & -4 \\ 2 & 10 \end{vmatrix} - 1\begin{vmatrix} -1 & -4 \\ 3 & 10 \end{vmatrix} + 2\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} \\
=& 1(0 \times 10 - (-4) \times 2) - 1(-1 \times 10 - (-4) \times 3) \\
& - 2(-1 \times 2 - 0 \times 3) \\
=& 1(8) - 1(2) + 2(-2) \\
=& 8 - 2 - 4 \\
=& 2.
\end{aligned}
$$

Since $\det(C)$ is non-zero, we have confirmed that C is invertible.

(d)

$$
\begin{aligned}
\det(D) =& \det\begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 5 & 10 & 5 \end{pmatrix} \\
=& 2\begin{vmatrix} 2 & 4 \\ 10 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & 4 \\ 5 & 5 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 5 & 10 \end{vmatrix} \\
=& 2(2 \times 5 - 4 \times 10) - 4(1 \times 5 - 4 \times 5) \\
& 6(1 \times 10 - 2 \times 5) \\
=& 2(-30) - 4(-15) + 6(0) \\
=& -60 + 60 + 0 \\
=& 0.
\end{aligned}
$$

Since $\det(D) = 0$, we have confirmed that D is not invertible.

(e)

$$\begin{aligned}\det(E) =& \det\begin{pmatrix} 1 & 2 & 4 \\ -2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ =& 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + 4\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} \\ =& 1(-1\times 1 - 1\times 1) - 2(-2\times 1 - 1\times 1) \\ & 4(-2\times 1 - (-1)\times 1) \\ =& 1(-2) - 2(-3) + 4(-1) \\ =& -2 + 6 - 4 \\ =& 0.\end{aligned}$$

Since $\det(E) = 0$, we have confirmed that E is not invertible.

(f)

$$\begin{aligned}\det(F) =& \det\begin{pmatrix} 1 & 1 & -4 \\ 2 & 1 & -6 \\ -3 & -1 & 9 \end{pmatrix} \\ =& 1\begin{vmatrix} 1 & -6 \\ -1 & 9 \end{vmatrix} - 1\begin{vmatrix} 2 & -6 \\ -3 & 9 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \\ =& 1(1\times 9 - (-6)\times(-1)) - 1(2\times 9 - (-6)\times(-3)) \\ & - 4(2\times(-1) - 1\times(-3)) \\ =& 1(3) - 1(0) - 4(1) \\ =& 3 - 0 - 4 \\ =& -1.\end{aligned}$$

Since $\det(F)$ is non-zero, we have confirmed that F is invertible.

(g)

$$\begin{aligned}\det(G) =& \det\begin{pmatrix} 1 & 4 & 1 \\ 1 & 6 & 3 \\ 2 & 3 & 0 \end{pmatrix} \\ =& 1\begin{vmatrix} 6 & 3 \\ 3 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ 2 & 3 \end{vmatrix} \\ =& 1(6\times 0 - 3\times 3) - 4(1\times 0 - 3\times 2) \\ & + 1(1\times 3 - 6\times 2) \\ =& 1(-9) - 4(-6) + 1(-9) \\ =& -9 + 24 - 9 \\ =& 6.\end{aligned}$$

Since $\det(G)$ is non-zero, we have confirmed that G is invertible.

(h)

$$\begin{aligned}
\det(H) =& \det\begin{pmatrix} 1 & -4 & 1 \\ 1 & 6 & -3 \\ -2 & 3 & 0 \end{pmatrix} \\
=& 1\begin{vmatrix} 6 & -3 \\ 3 & 0 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ -2 & 3 \end{vmatrix} \\
=& 1(6 \times 0 - (-3) \times 3) + 4(1 \times 0 - (-3) \times (-2)) \\
& + 1(1 \times 3 - 6 \times (-2)) \\
=& 1(9) + 4(-6) + 1(15) \\
=& 9 - 24 + 15 \\
=& 0.
\end{aligned}$$

Since $\det(H) = 0$, we have confirmed that H is not invertible.

8. (a) We have $\|(1,2)\| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5}$.
   Hence a unit vector in the direction of $(1,2)$ is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

   (b) We have $\|(-1,3)\| = \sqrt{(-1)^2 + 3^2} = \sqrt{1+9} = \sqrt{10}$.
   Hence a unit vector in the direction of $(-1,3)$ is $\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$.

   (c) We have $\|(1,0,-2,3)\| = \sqrt{1^2 + 0^2 + (-2)^2 + 3^2} = \sqrt{1+0+4+9} = \sqrt{14}$.
   Hence a unit vector in the direction of $(1,0,-2,3)$ is $\left(\frac{1}{\sqrt{14}}, 0, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

   (d) We have $\|(0,0,0)\| = \sqrt{0^2 + 0^2 + 0^2} = \sqrt{0+0+0} = \sqrt{0} = 0$.
   However $(0,0,0)$ doesn't have a direction, so it doesn't make any sense to ask for a vector of unit length in the same direction.

9. In this question we will use the formula $\theta = \cos^{-1}\left(\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|}\right)$, where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

   (a) The angle between $(1,2)$ and $(2,1)$ is

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{(1,2) \cdot (2,1)}{\|(1,2)\| \cdot \|(2,1)\|}\right) \\
&= \cos^{-1}\left(\frac{(1)(2) + (2)(1)}{\sqrt{1^2 + 2^2} \cdot \sqrt{2^2 + 1^2}}\right) \\
&= \cos^{-1}\left(\frac{4}{\sqrt{5} \cdot \sqrt{5}}\right) \\
&= \cos^{-1}\left(\frac{4}{5}\right) \\
&\simeq 0.64 \text{ to 2 d.p.}
\end{aligned}$$

(b) The angle between $(1,0,1)$ and $(-1,2,0)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(1,0,1)\cdot(-1,2,0)}{\|(1,0,1)\|\cdot\|(-1,2,0)\|}\right)\\ &= \cos^{-1}\left(\frac{(1)(-1)+(0)(2)+(1)(0)}{\sqrt{1^2+0^2+1^2}\cdot\sqrt{(-1)^2+2^2+0^2}}\right)\\ &= \cos^{-1}\left(\frac{-1}{\sqrt{2}\cdot\sqrt{5}}\right)\\ &= \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)\\ &\simeq 1.89 \text{ to 2 d.p.}\end{aligned}$$

Note that in this case the dot product between the two vectors is negative, so the angle between the vectors is greater than $\pi/2$.

(c) The angle between $(1,2,1)$ and $(-1,2,2)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(1,2,1)\cdot(-1,2,2)}{\|(1,2,1)\|\cdot\|(-1,2,2)\|}\right)\\ &= \cos^{-1}\left(\frac{(1)(-1)+(2)(2)+(1)(2)}{\sqrt{1^2+2^2+1^2}\cdot\sqrt{(-1)^2+2^2+2^2}}\right)\\ &= \cos^{-1}\left(\frac{5}{\sqrt{6}\cdot\sqrt{9}}\right)\\ &= \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)\\ &\simeq 0.82 \text{ to 2 d.p.}\end{aligned}$$

(d) The angle between $(1,1,2)$ and $(-1,-1,1)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(1,1,2)\cdot(-1,-1,1)}{\|(1,1,2)\|\cdot\|(-1,-1,1)\|}\right)\\ &= \cos^{-1}\left(\frac{(1)(-1)+(1)(-1)+(2)(1)}{\sqrt{1^2+1^2+2^2}\cdot\sqrt{(-1)^2+(-1)^2+1^2}}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{6}\cdot\sqrt{3}}\right)\\ &= \cos^{-1}(0)\\ &\simeq 1.57 \text{ to 2 d.p.}\end{aligned}$$

Note that in fact $\theta = \frac{\pi}{2}$ (The vectors are orthogonal, i.e., at right angles).

(e) This question doesn't make any sense since the zero vector has zero length. So it is not a line and it doesn't make an angle with any other vector.

10. (a)

$$
\begin{aligned}
(1,0,0)\times(0,1,0) =& \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\
=& \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\hat{k} \\
=&(0\times 0 - 0\times 1)\hat{\imath} - (1\times 0 - 0\times 0)\hat{\jmath} \\
&+ (1\times 1 - 0\times 0)\hat{k} \\
=&0\hat{\imath} - 0\hat{\jmath} + 1\hat{k} \\
=&\hat{k} \\
=&(0,0,1).
\end{aligned}
$$

(b)

$$
\begin{aligned}
(0,1,0)\times(1,0,0) =& \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\
=& \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\hat{k} \\
=&(1\times 0 - 0\times 0)\hat{\imath} - (0\times 0 - 0\times 1)\hat{\jmath} \\
&+ (0\times 0 - 1\times 1)\hat{k} \\
=&0\hat{\imath} - 0\hat{\jmath} - 1\hat{k} \\
=& -\hat{k} \\
=&(0,0,-1).
\end{aligned}
$$

Note this also follows from Part (a), since
$(0,1,0)\times(1,0,0) = -(1,0,0)\times(0,1,0) = -(0,0,1) = (0,0,-1)$

(c)

$$
\begin{aligned}
(1,2,-4)\times(-2,-4,8) =& \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -4 \\ -2 & -4 & -8 \end{vmatrix} \\
=& \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix}\hat{\imath} - \begin{vmatrix} 1 & -4 \\ -2 & 8 \end{vmatrix}\hat{\jmath} + \begin{vmatrix} 1 & 2 \\ -2 & -4 \end{vmatrix}\hat{k} \\
=&(2\times 8 - (-4)\times(-4))\hat{\imath} - (1\times 8 - (-4)\times(-2))\hat{\jmath} \\
&+ (1\times(-4) - 2\times(-2))\hat{k} \\
=&0\hat{\imath} - 0\hat{\jmath} + 0\hat{k} \\
=&(0,0,0).
\end{aligned}
$$

Note that $(-2,-4,8) = -2(1,2,-4)$, so the original vectors are parallel and it follows that the cross product is the zero vector.

(d)

$$\begin{aligned}
(1,2,-1)\times(-1,2,3) =& \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} \\
=& \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \hat{k} \\
=& (2\times 3 - (-1)\times 2)\hat{\imath} - (1\times 3 - (-1)\times(-1))\hat{\jmath} \\
& + (1\times 2 - 2\times(-1))\hat{k} \\
=& 8\hat{\imath} - 2\hat{\jmath} + 4\hat{k} \\
=& (8,-2,4).
\end{aligned}$$

(e) This question does not make any sense, we can only take the cross product of vectors in $\mathbb{R}^3$. However note that

$$\begin{aligned}
(1,2,0)\times(2,1,0) =& \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} \\
=& \begin{vmatrix} 2 & 0 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \hat{k} \\
=& (2\times 0 - 0\times 1)\hat{\imath} - (1\times 0 - 0\times 2)\hat{\jmath} \\
& + (1\times 1 - 2\times 2)\hat{k} \\
=& 0\hat{\imath} - 0\hat{\jmath} - 3\hat{k} \\
=& (0,0,-3).
\end{aligned}$$

11. (a) We have

$$\begin{aligned}
\det\left[\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \\
&= (1-\lambda)(2-\lambda) - 2(3) \\
&= \lambda^2 - 3\lambda + 2 - 6 \\
&= \lambda^2 - 3\lambda - 4.
\end{aligned}$$

Hence the characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda+1)(\lambda-4) = 0$. Thus the eigenvalues are $\lambda = -1$ and $\lambda = 4$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -1$:
We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} x+2y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$. Hence we have the equations $x+2y=-x$ and $3x+2y=-y$. Both these equations reduce to $x=-y$, so taking $y=1$, say, we obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\lambda = 4$:

We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} x+2y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$. Hence we have the equations $x + 2y = 4x$ and $3x + 2y = 4y$. Both these equations reduce to $3x = 2y$, so taking $y = 3$, say, we obtain the eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

(b) We have

$$\begin{aligned} \det\left[\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} -1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} \\ &= (-1-\lambda)(-\lambda) - 2(1) \\ &= \lambda^2 + \lambda - 2. \end{aligned}$$

Hence the characteristic equation is $\lambda^2 + \lambda - 2 = 0$ or $(\lambda + 2)(\lambda - 1) = 0$. Thus the eigenvalues are $\lambda = -2$ and $\lambda = 1$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -2$:

We have $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -x+2y \\ x \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$. Hence we have the equations $-x + 2y = -2x$ and $x = -2y$. Both these equations reduce to $x = -2y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

$\lambda = 1$:

We have $\begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -x+2y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$. Hence we have the equations $-x + 2y = x$ and $x = y$. Both these equations reduce to $x = y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) We have

$$\begin{aligned} \det\left[\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{pmatrix} \\ &= (-2-\lambda)(5-\lambda) - 3(-2) \\ &= \lambda^2 - 3\lambda - 10 + 6 \\ &= \lambda^2 - 3\lambda - 4. \end{aligned}$$

Hence the characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda + 1)(\lambda - 4) = 0$. Thus the eigenvalues are $\lambda = -1$ and $\lambda = 4$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -1$:
We have $\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -2x+3y \\ -2x+5y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.
Hence we have the equations $-2x + 3y = -x$ and $-2x + 5y = -y$. Both these equations reduce to $x = 3y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$\lambda = 4$:
We have $\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -2x+3y \\ -2x+5y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$.
Hence we have the equations $-2x + 3y = 4x$ and $-2x + 5y = 4y$. Both these equations reduce to $y = 2x$, so taking $x = 1$, say, we obtain the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) We have

$$\begin{aligned} \det\left[\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{pmatrix} \\ &= (2-\lambda)(3-\lambda) - 0(0) \\ &= (2-\lambda)(3-\lambda). \end{aligned}$$

Hence the characteristic equation is $(2 - \lambda)(3 - \lambda) = 0$ and the eigenvalues are $\lambda = 2$ and $\lambda = 3$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = 2$:
We have $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 2x \\ 3y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$. Hence we have the equations $2x = 2x$ and $3y = 2y$. The first of these equations places no restriction on x and the second yields $y = 0$. Thus taking $x = 1$, say, we obtain the eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\lambda = 3$:
We have $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 2x \\ 3y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$. Hence we have the equations $2x = 3x$ and $3y = 3y$. The first of these equations yields $x = 0$ and the second places no restriction on y. Thus taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(e) We have

$$\det\left[\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}$$
$$= (3-\lambda)(1-\lambda) - (-1)(1)$$
$$= \lambda^2 - 4\lambda + 3 + 1$$
$$= \lambda^2 - 4\lambda + 4.$$

Hence the characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda^2 - 2)^2 = 0$ and we have a repeated eigenvalue $\lambda = 2$.
We will now find the eigenvector(s) corresponding to this eigenvalue.

We have $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 3x-y \\ x+y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$. Hence we have the equations $3x - y = 2x$ and $x + y = 2y$. Both these equations reduce to $x = y$, so taking $y = 1$, say, we obtain the single eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (this matrix has a defect).

(f) We have

$$\det\left[\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix}$$
$$= (-2-\lambda)(-2-\lambda) - (0)(0)$$
$$= (-2-\lambda)^2.$$

Hence the characteristic equation is $(-2-\lambda)^2 = 0$ and we have a repeated eigenvalue $\lambda = -2$.
We will now find the eigenvector(s) corresponding to this eigenvalue.

We have $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -2x \\ -2y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$. Hence we have the equations $-2x = -2x$ and $-2y = -2y$. Neither of these equations place any restrictions on either x or y. Thus in this case every non-zero vector is an eigenvector corresponding to the eigenvalue $\lambda = -2$.