Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Chapter 3 Solutions

1. (a) Since $f(x) = 5$ is a constant, $f'(x) = 0$.

   (b) Since $f(x) = -\pi\cos(e)$ is a constant, $f'(x) = 0$.

   (c) Here f is of the form $f(x) = x^n$, with $n = 2$.
   Thus $f'(x) = 2x^{2-1} = 2x^1 = 2x$.

   (d) Here f is of the form $f(x) = x^n$, with $n = \frac{9}{2}$.
   Thus $f'(x) = \frac{9}{2}x^{\frac{9}{2}-1} = \frac{9}{2}x^{\frac{7}{2}}$.

   (e) Here f is of the form $f(x) = x^n$, with $n = -5$.
   Thus $f'(x) = -5x^{-5-1} = -5x^{-6}$.

   (f) Here f is of the form $f(x) = x^n$, with $n = \cos(2)$.
   Thus $f'(x) = \cos(2)x^{\cos(2)-1}$.

   (g) Here f is of the form $f(x) = e^{ax}$, with $a = 4$.
   Thus $f'(x) = 4e^{4x}$.

   (h) Here f is of the form $f(x) = e^{ax}$, with $a = \frac{3}{2}$.
   Thus $f'(x) = \frac{3}{2}e^{\frac{3}{2}x}$.

   (i) Here f is of the form $f(x) = e^{ax}$, with $a = -6$.
   Thus $f'(x) = -6e^{-6x}$.

   (j) Here f is of the form $f(x) = e^{ax}$, with $a = \pi$.
   Thus $f'(x) = \pi e^{\pi x}$.

   (k) Here f is of the form $f(x) = \ln(ax)$, with $a = 4$.
   Thus $f'(x) = \frac{1}{x}$.

   (l) Here f is of the form $f(x) = \ln(ax)$, with $a = -\pi$.
   Thus $f'(x) = \frac{1}{x}$.

   (m) Here f is of the form $f(x) = \ln(ax)$, with $a = \frac{1}{2}$.
   Thus $f'(x) = \frac{1}{x}$.

   (n) Here f is of the form $f(x) = \sin(ax)$, with $a = 2$.
   Thus $f'(x) = 2\cos(2x)$.

   (o) Here f is of the form $f(x) = \sin(ax)$, with $a = -2$.
   Thus $f'(x) = -2\cos(-2x)$.

(p) Here f is of the form $f(x) = \sin(ax)$, with $a = e$.
Thus $f'(x) = e\cos(ex)$.

(q) Here f is of the form $f(x) = \cos(ax)$, with $a = 3$.
Thus $f'(x) = -3\sin(3x)$.

(r) Here f is of the form $f(x) = \cos(ax)$, with $a = -3$.
Thus $f'(x) = -(-3)\sin(-3x) = 3\sin(-3x)$.

(s) Here f is of the form $f(x) = \cos(ax)$, with $a = -\pi$.
Thus $f'(x) = -(-\pi)\sin(-\pi x) = \pi\sin(-\pi x)$.

2. (a) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1) + \frac{d}{dx}(3x) + \frac{d}{dx}\left(-2x^2\right) + \frac{d}{dx}\left(3x^3\right) + \frac{d}{dx}\left(-4x^4\right) \\ &= \frac{d}{dx}(1) + 3\frac{d}{dx}(x) - 2\frac{d}{dx}\left(x^2\right) + 3\frac{d}{dx}\left(x^3\right) - 4\frac{d}{dx}\left(x^4\right) \\ &= 0 + 3(1) - 2(2x) + 3(3x^2) - 4(4x^3) \\ &= 3 - 4x + 9x^2 - 16x^3. \end{aligned}$$

Note that in your assignment or exam solutions you don't need to give as much detail as this. I am just setting out everything carefully until you get used to the ideas involved.

(b) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-x^{-1}\right) + \frac{d}{dx}(2\sin 4x) \\ &= -\frac{d}{dx}\left(x^{-1}\right) + 2\frac{d}{dx}(\sin 4x) \\ &= -\left(-x^{-2}\right) + 2(4\cos(4x)) \\ &= x^{-2} + 8\cos(4x). \end{aligned}$$

(c) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(3e^{-\frac{1}{2}x}\right) + \frac{d}{dx}\left(-2\cos\left(\frac{1}{2}x\right)\right) \\ &= 3\frac{d}{dx}\left(e^{-\frac{1}{2}x}\right) - 2\frac{d}{dx}\left(\cos\left(\frac{1}{2}x\right)\right) \\ &= 3\left(-\frac{1}{2}e^{-\frac{1}{2}x}\right) - 2\left(-\frac{1}{2}\sin\left(\frac{1}{2}x\right)\right) \\ &= -\frac{3}{2}e^{-\frac{1}{2}x} + \sin\left(\frac{1}{2}x\right). \end{aligned}$$

(d) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}(2\ln(-x)) + \frac{d}{dx}(4\cos(-3x)) + \frac{d}{dx}\left(-e^{-\frac{3}{2}x}\right) \\ &= 2\frac{d}{dx}(\ln(-x)) + 4\frac{d}{dx}(\cos(-3x)) - \frac{d}{dx}\left(e^{-\frac{3}{2}x}\right) \\ &= 2\left(\frac{1}{x}\right) + 4(3\sin(-3x)) - \left(-\frac{3}{2}e^{-\frac{3}{2}x}\right) \\ &= \frac{2}{x} + 12\sin(-3x) + \frac{3}{2}e^{-\frac{3}{2}x}. \end{aligned}$$

(e) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-2x^2\right) + \frac{d}{dx}\left(3\ln(3x)\right) + \frac{d}{dx}\left(e^{\cos(1)x}\right) \\ &= -2\frac{d}{dx}\left(x^2\right) + 3\frac{d}{dx}\left(\ln(3x)\right) + \frac{d}{dx}\left(e^{\cos(1)x}\right) \\ &= -2\left(2x\right) + 3\left(\frac{1}{x}\right) + \cos(1)e^{\cos(1)x} \\ &= -4x + \frac{3}{x} + \cos(1)e^{\cos(1)x}. \end{aligned}$$

(f) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2\sin(3x)\right) + \frac{d}{dx}\left(-3\sin(2x)\right) + \frac{d}{dx}\left(2\cos(3x)\right) + \frac{d}{dx}\left(-3\cos(2x)\right) \\ &= 2\frac{d}{dx}\left(\sin(3x)\right) - 3\frac{d}{dx}\left(\sin(2x)\right) + 2\frac{d}{dx}\left(\cos(3x)\right) - 3\frac{d}{dx}\left(\cos(2x)\right) \\ &= 2\left(3\cos(3x)\right) - 3\left(2\cos(2x)\right) + 2\left(-3\sin(3x)\right) - 3\left(-2\sin(2x)\right) \\ &= 6\cos(3x) - 6\cos(2x) - 6\sin(3x) + 6\sin(2x) \\ &= 6(\cos(3x) - \cos(2x) - \sin(3x) + \sin(2x)). \end{aligned}$$

(g) Using the sum rule,

$$f'(x) = \frac{d}{dx}\left(e^2 - 4\right) + \frac{d}{dx}\left(e^{2x}\right) = 0 + 2e^{2x} = 2e^{2x}.$$

Note that here we didn't need the multiple rule and also we were able to deal with the two terms e^2 and -4 all at once since $e^2 - 4$ is just a constant.

(h) Using the sum and multiple rules,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-3x^{-3}\right) + \frac{d}{dx}\left(4x^4\right) + \frac{d}{dx}\left(5x^{-5}\right) + \frac{d}{dx}(3) \\ &= -3\frac{d}{dx}\left(x^{-3}\right) + 4\frac{d}{dx}\left(x^4\right) + 5\frac{d}{dx}\left(x^{-5}\right) + \frac{d}{dx}(3) \\ &= -3\left(-3x^{-4}\right) + 4\left(4x^3\right) + 5\left(-5x^{-6}\right) + 0 \\ &= 9x^{-4} + 16x^3 + -25x^{-6}. \end{aligned}$$

Note that $3x^0$ is just the number 3 (unless $x = 0$ when x^0 is not defined), so it differentiates to zero. We could also obtain the derivative as $3\left(0x^{-1}\right) = 0$ but it would be a bit more work.

For the remainder of the questions we will still need the sum and multiple rules but I will not mention them explicitly.

3. (a) Here we have to use the product rule twice. First let us differentiate $g(x) = 3x^2\sin(2x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(3x^2\right)\sin(2x) + 3x^2\frac{d}{dx}\left(\sin(2x)\right) \\ &= 6x\sin(2x) + 3x^2\left(2\cos(2x)\right) \\ &= 6x\sin(2x) + 6x^2\cos(2x). \end{aligned}$$

Next we differentiate $h(x) = -e^{2x}\cos(x)$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(-e^{2x}\right)\cos(x) - e^{2x}\frac{d}{dx}\left(\cos(x)\right) \\ &= -2e^{2x}\cos(x) - e^{2x}(-\sin(x)) \\ &= -2e^{2x}\cos(x) + e^{2x}\sin(x). \end{aligned}$$

Putting this together we obtain

$$f'(x) = g'(x) + h'(x) = 6x\sin(2x) + 6x^2\cos(2x) - 2e^{2x}\cos(x) + e^{2x}\sin(x).$$

(b) Again we have to use the product rule twice. First let us differentiate $g(x) = (2x^2 - 3x^{-3} + 5x^4)\sin(-5x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(2x^2 - 3x^{-3} + 5x^4\right)\sin(-5x) + (2x^2 - 3x^{-3} + 5x^4)\frac{d}{dx}\left(\sin(-5x)\right) \\ &= \left(4x + 9x^{-4} + 20x^3\right)\sin(-5x) + (2x^2 - 3x^{-3} + 5x^4)\left(-5\cos(-5x)\right) \\ &= \left(4x + 9x^{-4} + 20x^3\right)\sin(-5x) - 5(2x^2 - 3x^{-3} + 5x^4)\cos(-5x). \end{aligned}$$

Next we differentiate $h(x) = e^{-3x}(\sin(2x) - \cos(x))$.

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(e^{-3x}\right)(\sin(2x) - \cos(x)) + e^{-3x}\frac{d}{dx}\left(\sin(2x) - \cos(x)\right) \\ &= -3e^{-3x}(\sin(2x) - \cos(x)) + e^{-3x}(2\cos(2x) + \sin(x)). \end{aligned}$$

Putting this together we obtain

$$\begin{aligned} f'(x) =& g'(x) + h'(x) \\ =& (4x + 9x^{-4} + 20x^3)\sin(-5x) - 5(2x^2 - 3x^{-3} + 5x^4)\cos(-5x) \\ & - 3e^{-3x}(\sin(2x) - \cos(x)) + e^{-3x}(2\cos(2x) + \sin(x)). \end{aligned}$$

(c) Here there is a quick method and a slow method.
The quick method is to spot

$$f(x) = \sin(x)\sin(x) + \cos(x)\cos(x) = \sin^2(x) + \cos^2(x) = 1,$$

so that $f'(x) = 0$.
For the slow method, we use the product rule twice:

$$\begin{aligned} f'(x)) =& \frac{d}{dx}\left(\sin(x)\right)\sin(x) + \sin(x)\frac{d}{dx}\left(\sin(x)\right) \\ & + \frac{d}{dx}\left(\cos(x)\right)\cos(x) + \cos(x)\frac{d}{dx}\left(\cos(x)\right) \\ =& \cos(x)\sin(x) + \sin(x)\cos(x) - \sin(x)\cos(x) + \cos(x)(-\sin(x)) \\ =& 0, \text{ as expected.} \end{aligned}$$

Note that when we study the chain rule we will see another way of differentiating functions such as $\sin^2(x)$ and $\cos^2(x)$.

(d) Here there are also two methods.
For the first method we will use the half angle trig formulae to obtain

$$f(x) = \sin^2(x) - \cos^2(x) = \frac{1-\cos(2x)}{2} - \frac{1+\cos(2x)}{2} = -\cos(2x),$$

so that $f'(x) = 2\sin(2x)$.
For the second method we use the product rule twice:

$$\begin{aligned} f'(x)) =& \frac{d}{dx}(\sin(x))\sin(x) + \sin(x)\frac{d}{dx}(\sin(x)) \\ &- \left[\frac{d}{dx}(\cos(x))\cos(x) + \cos(x)\frac{d}{dx}(\cos(x))\right] \\ =& \cos(x)\sin(x) + \sin(x)\cos(x) - [-\sin(x)\cos(x) + \cos(x)(-\sin(x))] \\ =& 4\sin(x)\cos(x). \end{aligned}$$

This 'looks' different from our first answer but it is in fact the same, since using the double angle trig formula, we obtain

$$2\sin(2x) = 2(2\sin(x)\cos(x)) = 4\sin(x)\cos(x).$$

This illustrates an important general point. If you use two different methods to solve a problem, you might end up with answers that look different but are in fact the same.

(e) Here we just have to use one application of the product rule to differentiate $g(x) = e^{2x}\ln(2x)$.

$$\begin{aligned} g'(x)) =& \frac{d}{dx}\left(e^{2x}\right)\ln(2x) + e^{2x}\frac{d}{dx}(\ln(x))) \\ =& 2e^{2x}\ln(2x) + e^{2x}\left(\frac{1}{x}\right) \\ =& 2e^{2x}\ln(2x) + \frac{e^{2x}}{x}. \end{aligned}$$

Since the derivative of $h(x) = -x^2 + x^3 + 1$ is $h'(x) = -2x + 3x^2$, we have

$$f'(x) = g'(x) + h'(x) = 2e^{2x}\ln(2x) + \frac{e^{2x}}{x} - 2x + 3x^2.$$

(f) Here we don't need to use the product rule at all. Since $e^0 = 1$ and $\ln(1) = 0$, we have $f(x) = \cos(2x)$, so that $f'(x) = -2\sin(2x)$.
It is very important to keep your eye out for situations like this, where an initial simplification makes the differentiation a lot easier.

(g) Here we have a product of three terms, so things are a little more complicated. What we will do is to use the product rule to differentiate a product two of the terms and then use the product rule again.
First we will differentiate $g(x) = x^2e^{3x}$.

$$g'(x) = \frac{d}{dx}\left(x^2\right)e^{3x} + x^2\frac{d}{dx}\left(e^{3x}\right) = 2xe^{3x} + x^2\left(3e^{3x}\right) = 2xe^{3x} + 3x^2e^{3x}.$$

We can now differentiate f by writing it as $f(x) = (x^2e^{3x})(\cos(4x))$.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^2e^{3x}\right)\cos(4x) + x^2e^{3x}\frac{d}{dx}(\cos(4x)) \\ &= \left(2xe^{3x} + 3x^2e^{3x}\right)\cos(4x) + x^2e^{3x}(-4\sin(4x)) \\ &= (2xe^{3x} + 3x^2e^{3x})\cos(4x) - 4x^2e^{3x}\sin(4x). \end{aligned}$$

4. (a) Here we use the quotient rule. If $\sin(x) \neq 0$, we have

$$f'(x) = \frac{\frac{d}{dx}(x)\sin(x) - x\frac{d}{dx}(\sin(x))}{\sin^2(x)} = \frac{\sin(x) - x\cos(x)}{\sin^2(x)}.$$

(b) By the quotient rule, we have

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(2e^{3x})(x^2+x+1) - 2e^{3x}\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2} \\ &= \frac{6e^{3x}(x^2+x+1) - 2e^{3x}(2x+1)}{(x^2+x+1)^2}. \end{aligned}$$

(c) In this case we will first have to use the product rule to differentiate $g(x) = x\cos(2x)$ and then use the quotient rule. Now

$$\begin{aligned} g'(x) &= \frac{d}{dx}(x)\cos(2x) + x\frac{d}{dx}(\cos(2x)) \\ &= 1\cos(2x) + x(-2\sin(2x)) \\ &= \cos(2x) - 2x\sin(2x). \end{aligned}$$

We can now use the quotient rule to obtain (when $x > \frac{1}{2}$)

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(x\cos(2x))\ln(2x) - x\cos(2x)\frac{d}{dx}(\ln(2x))}{(\ln(2x))^2} \\ &= \frac{(\cos(2x) - 2x\sin(2x))\ln(2x) - x\cos(2x)\left(\frac{1}{x}\right)}{(\ln(2x))^2} \\ &= \frac{(\cos(2x) - 2x\sin(2x))\ln(2x) - \cos(2x)}{(\ln(2x))^2}. \end{aligned}$$

(d) Here we will have to use the product rule for the numerator and the denomonator before we use the quotient rule.
First we will differentiate $g(x) = (x^3 - 2x)\cos(-2x)$ using the product rule.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(x^3 - 2x\right)\cos(-2x) + (x^3 - 2x)\frac{d}{dx}(\cos(-2x)) \\ &= \left(3x^2 - 2\right)\cos(-2x) + (x^3 - 2x)(2\sin(-2x)) \\ &= \left(3x^2 - 2\right)\cos(-2x) + 2(x^3 - 2x)\sin(-2x). \end{aligned}$$

Next we will differentiate $h(x) = \sin(x)\cos(x)$ using the product rule. Note we could also use $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$ in which case the answer below would look different but would be the same.

$$\begin{aligned} h'(x) &= \frac{d}{dx}(\sin(x))\cos(x) + \sin(x)\frac{d}{dx}(\cos(x)) \\ &= \cos(x)\cos(x) + \sin(x)\frac{d}{dx}(-\sin(x)) \\ &= \cos^2(x) - \sin^2(x). \end{aligned}$$

We can now use the quotient rule to differentiate f (when $\sin(x)\cos(x) \neq 0$).

$$\begin{aligned} f'(x) =& \frac{\frac{d}{dx}((x^3 - 2x)\cos(-2x))\sin(x)\cos(x)}{(\sin(x)\cos(x))^2} \\ &- \frac{(x^3 - 2x)\cos(-2x)\frac{d}{dx}(\sin(x)\cos(x))}{(\sin(x)\cos(x))^2} \\ =& \frac{((3x^2 - 2)\cos(-2x) + 2(x^3 - 2x)\sin(-2x))\sin(x)\cos(x)}{\sin^2(x)\cos^2(x)} \\ &- \frac{((x^3 - 2x)\cos(-2x))(\cos^2(x) - \sin^2(x))}{\sin^2(x)\cos^2(x)} \end{aligned}$$

(e) By the quotient rule, we have (when $\sin(x) \neq 0$)

$$\begin{aligned} f'(x) &= \frac{\frac{d}{dx}(\cos(x))\sin(x) - \cos(x)\frac{d}{dx}(\sin(x))}{(\sin(x))^2} \\ &= \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= -\frac{1}{\sin^2(x)} \quad (\text{using } \sin^2(x) + \cos^2(x) = 1) \\ &= -\operatorname{cosec}^2(x). \end{aligned}$$

(f) Before we use the quotient rule here, we will have to use the product rule to differentiate $g(x) = 2\cos(3x)\ln(3x)$. When $x > 0$,

$$\begin{aligned} g'(x) &= \frac{d}{dx}(2\cos(3x))\ln(3x) + 2\cos(3x)\frac{d}{dx}(\ln(3x)) \\ &= -6\sin(3x)\ln(3x) + 2\cos(3x)\left(\frac{1}{x}\right) \\ &= -6\sin(3x)\ln(3x) + \frac{2\cos(3x)}{x}. \end{aligned}$$

Now using the quotient rule, when $x > 0$, we have

$$f'(x) = \frac{\frac{d}{dx}\left(2\cos(3x)\ln(3x) + x^2\right)\left(2e^{-x} + x^4\right)}{\left(2e^{-x} + x^4\right)^2} - \frac{\left(2\cos(3x)\ln(3x) + x^2\right)\frac{d}{dx}\left(2e^{-x} + x^4\right)}{\left(2e^{-x} + x^4\right)^2}$$

$$= \frac{\left(-6\sin(3x)\ln(3x) + \dfrac{2\cos(3x)}{x} + 2x\right)\left(2e^{-x} + x^4\right)}{(2e^{-x} + x^4)^2} - \frac{(2\cos(3x)\ln(3x) + x^2)(-2e^{-x} + 4x^3)}{(2e^{-x} + x^4)^2}.$$

5. (a) Here we will use the chain rule with $u = x^2$ and $y = f(x) = e^u$. Then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u\,(2x) = 2xe^{x^2}.$$

(b) Here we will use the chain rule with $u = 2x^2 + 3x$ and $y = f(x) = \sin(u)$. Then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u)\,(4x + 3) = (4x + 3)\cos\left(2x^2 + 3x\right).$$

(c) Here we will use the chain rule with $u = \sin(x)$ and $y = f(x) = \ln(u)$. Then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u}\cos(x) = \frac{\cos(x)}{\sin(x)} = \cot(x).$$

(d) Before we use the chain rule here, we will first use the product rule to differentiate $u = xe^x$. We have

$$\frac{du}{dx} = \frac{d}{dx}(x)\,e^x + x\frac{d}{dx}(e^x) = (1)e^x + xe^x = e^x + xe^x.$$

We can now use the chain rule with $u = xe^x$ and $y = f(x) = \cos(u)$. Then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u)\,(e^x + xe^x) = -(e^x + xe^x)\sin(xe^x).$$

(e) Here we will use the chain rule with $u = \sin(x)$ and $y = f(x) = \sin(u)$. Then

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos(u)\cos(x) = \cos(\sin(x))\cos(x) = \cos(x)\cos(\sin(x)).$$

(f) Here we will use the chain rule twice. We will first use it to differentiate $u = \sin(\cos(x))$. To do this we let $v = \cos(x)$ and $u = \sin(v)$. Then

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \cos(v)(-\sin(x)) = -\sin(x)\cos(v) = -\sin(x)\cos(\cos(x)).$$

Next we will use the chain rule again, this time with $u = \sin(\cos(x))$ and $y = f(x) = \cos(u)$.

$$\begin{aligned} f'(x) &= \frac{dy}{dx} \\ &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\sin(u)(-\sin(x)\cos(\cos(x))) \\ &= \sin(\sin(\cos(x)))\sin(x)\cos(\cos(x)) \\ &= \sin(x)\cos(\cos(x))\sin(\sin(\cos(x))). \end{aligned}$$

(g) Again we will use the chain rule twice. We will first use it to differentiate $u = \sin(x^2)$. To do this we let $v = x^2$ and $u = \sin(v)$. Then

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \cos(v)(2x) = 2x\cos(x^2).$$

We will now use the chain rule again, this time with $u = \sin(x^2)$ and $y = f(x) = e^u$.

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u(2x\cos(x^2)) = 2x\cos(x^2)e^{\sin(x^2)}.$$

6. In all these questions we will differentiate $f(x)$ and solve the equation $f'(x) = 0$ to find the critical points.

(a) In this case $f'(x) = 3x^2 - 6x - 9$. Thus we have to solve the equation

$$\begin{aligned} 3x^2 - 6x - 9 = 0 &\Leftrightarrow x^2 - 2x - 3 = 0 \\ &\Leftrightarrow (x+1)(x-3) = 0 \\ &\Leftrightarrow x = -1 \text{ or } x = 3. \end{aligned}$$

Thus the critical points are $x = -1$ and $x = 3$.

(b) In this case $f'(x) = 3x^2 + 6x$. Thus we have to solve the equation

$$\begin{aligned} 3x^2 + 6x = 0 &\Leftrightarrow x^2 + 2x = 0 \\ &\Leftrightarrow (x+2)x = 0 \\ &\Leftrightarrow x = -2 \text{ or } x = 0. \end{aligned}$$

Thus the critical points are $x = -2$ and $x = 0$.

(c) In this case $f'(x) = -6x^2 - 18x + 24$. Thus we have to solve the equation

$$\begin{aligned} -6x^2 - 18x + 24 = 0 &\Leftrightarrow x^2 + 3x - 4 = 0 \\ &\Leftrightarrow (x+4)(x-1) = 0 \\ &\Leftrightarrow x = -4 \text{ or } x = 1. \end{aligned}$$

Thus the critical points are $x = -4$ and $x = 1$.

(d) In this case $f'(x) = 6x^2 + 6x + 6$. Thus we have to solve the equation

$$6x^2 + 6x + 6 = 0 \Leftrightarrow x^2 + x + 1 = 0.$$

However the equation $x^2 + x + 1 = 0$ has no real solutions, so there are no critical points.

(e) In this case $f'(x) = -3e^{-3x} + 7$. Thus we have to solve the equation

$$\begin{aligned}
-3e^{-3x} + 7 = 0 &\Leftrightarrow -3e^{-3x} = -7 \\
&\Leftrightarrow e^{-3x} = \frac{7}{3} \\
&\Leftrightarrow \ln\left(e^{-3x}\right) = \ln\left(\frac{7}{3}\right) \\
&\Leftrightarrow -3x = \ln\left(\frac{7}{3}\right) \\
&\Leftrightarrow x = -\frac{1}{3}\ln\left(\frac{7}{3}\right).
\end{aligned}$$

So there is one critical point of f, that is $x = -\frac{1}{3}\ln\left(\frac{7}{3}\right)$.

(f) In this case $f'(x) = 4e^{4x} = 5$. Thus we have to solve the equation

$$4e^{4x} = 5 \Leftrightarrow 4e^{4x} = -5.$$

However $4e^{4x} > 0$, so that $e^{4x} = -5$ and hence $4e^{4x} + 5 = 0$ have no solutions. Thus f has no critical points.

(g) In this case $f'(x) = \cos(x)$. Thus we have to solve the equation

$$\cos(x) = 0 \Leftrightarrow x = \frac{\pi}{2} + k\pi \text{ for each } k \in \mathbb{Z}.$$

Thus the critical points are $x = \frac{\pi}{2} + k\pi$ for each $k \in \mathbb{Z}$.

(h) In this case $f'(x) = -2\sin(2x)$. Thus we have to solve the equation

$$\begin{aligned}
-2\sin(2x) = 0 &\Leftrightarrow \sin(2x) = 0 \\
&\Leftrightarrow 2x = k\pi \text{ for each } k \in \mathbb{Z} \\
&\Leftrightarrow x = \frac{k\pi}{2} \text{ for each } k \in \mathbb{Z}
\end{aligned}$$

Thus the critical points are $x = \frac{k\pi}{2}$ for each $k \in \mathbb{Z}$.

7. In these questions we will find where the global maxima and minima of each of the following functions occur by evaluating the functions at the endpoints of the domain and at any critical points that lie in the domain.

(a) Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 - 3x^2 - 9x + 12 \end{aligned}$$

have already been found in Question 6(a): they are $x = -1$ and $x = 3$.

(i) In this case we evaluate $f(x)$ at $x = -4, -1, 3, 6$. $f(-4) = -64$, $f(-1) = 17$, $f(3) = -15$ and $f(6) = 66$.
Hence the global maximum of f is 66 attained at $x = 6$ and the global minimum of f is -64 attained at $x = -4$.

(ii) In this case we evaluate $f(x)$ at $x = -3, -1, 3, 5$. $f(-3) = -15$, $f(-1) = 17$, $f(3) = -15$ and $f(5) = 17$.
Hence the global maximum of f is 17 attained at $x = -1$ and $x = 5$, and the global minimum of f is -15 attained at $x = -3$ and $x = 3$.

(iii) In this case we evaluate $f(x)$ at $x = -2, -1, 3, 4$. $f(-2) = 10$, $f(-1) = 17$, $f(3) = -15$ and $f(4) = -8$.
Hence the global maximum of f is 17 attained at $x = -1$ and the global minimum of f is -15 attained at $x = 3$.

(iv) In this case we evaluate $f(x)$ at $x = 0, 3$. $f(0) = 12$ and $f(3) = -15$.
Hence the global maximum of f is 12 attained at $x = 0$ and the global minimum of f is -15 attained at $x = 3$.

(b) Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x^3 + 3x^2 - 5 \end{aligned}$$

have already been found in Question 6(b): they are $x = -2$ and $x = 0$.

(i) In this case we evaluate $f(x)$ at $x = -3, -2, 0, 1$. $f(-3) = -5$, $f(-2) = -1$, $f(0) = -5$ and $f(1) = -1$.
Hence the global maximum of f is -1 attained at $x = -2$ and $x = 1$, and the global minimum of f is -5 attained at $x = -3$ and $x = 0$.

(ii) In this case we evaluate $f(x)$ at $x = -3, -2, 0$. $f(-3) = -5$, $f(-2) = -1$ and $f(0) = -5$.
Hence the global maximum of f is -1 attained at $x = -2$ and the global minimum of f is -5 attained at $x = -3$ and $x = 0$.

(iii) In this case we evaluate $f(x)$ at $x = -2, 0, 1$. $f(-2) = -1$, $f(0) = -5$ and $f(1) = -1$.
Hence the global maximum of f is -1 attained at $x = -2$ and $x = 1$, and the global minimum of f is -5 attained at $x = 0$.

(iv) In this case we evaluate $f(x)$ at $x = -1, 0$. $f(-1) = -3$ and $f(0) = -5$.
Hence the global maximum of f is -3 attained at $x = -1$ and the global minimum of f is -5 attained at $x = 0$.

(c) Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto -2x^3 - 9x^2 + 24x - 1 \end{aligned}$$

have already been found in Question 6(c): they are $x = -4$ and $x = 1$.

(i) In this case we evaluate $f(x)$ at $x = -7, -4, 1, 3$. $f(-7) = 76$, $f(-4) = -113$, $f(1) = 12$ and $f(3) = -64$.
Hence the global maximum of f is 76 attained at $x = -7$ and the global minimum of f is -113 attained at $x = -4$.

(ii) In this case we evaluate $f(x)$ at $x = -6, -4, 1, 4$. $f(-6) = -37$, $f(-4) = -113$, $f(1) = 12$ and $f(4) = -177$.
Hence the global maximum of f is 12 attained at $x = 1$ and the global minimum of f is -177 attained at $x = 4$.

(iii) In this case we evaluate $f(x)$ at $x = -4, 1$. $f(-4) = -113$ and $f(1) = 12$.
Hence the global maximum of f is 12 attained at $x = 1$ and the global minimum of f is -113 attained at $x = -4$.

(iv) In this case we evaluate $f(x)$ at $x = 2, 3$. $f(2) = -5$ and $f(3) = -64$.
Hence the global maximum of f is -5 attained at $x = 2$ and the global minimum of f is -64 attained at $x = 3$.

(d) Note that we saw in Question 6(d) that the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto 2x^3 + 3x^2 + 6x + 5 \end{aligned}$$

has no critical points.

(i) In this case we evaluate $f(x)$ at $x = -1, 1$. $f(-1) = 0$ and $f(1) = 16$.
Hence the global maximum of f is 16 attained at $x = 1$ and the global minimum of f is 0 attained at $x = -1$.

(ii) In this case we evaluate $f(x)$ at $x = -1, 3$. $f(-1) = 0$ and $f(3) = 104$.
Hence the global maximum of f is 104 attained at $x = 3$ and the global minimum of f is 0 attained at $x = -1$.

(iii) In this case we evaluate $f(x)$ at $x = -4, 1$. $f(-4) = -99$ and $f(1) = 16$.
Hence the global maximum of f is 16 attained at $x = 1$ and the global minimum of f is -99 attained at $x = -4$.

(iv) In this case we evaluate $f(x)$ at $x = -4, 3$. $f(-4) = -99$ and $f(3) = 104$.
Hence the global maximum of f is 104 attained at $x = 3$ and the global minimum of f is -99 attained at $x = -4$.

(e) Note that the critical point of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{-3x} + 7x \end{aligned}$$

has already been found in Question 6(e): it is $x = -\frac{1}{3}\ln\left(\frac{7}{3}\right)$.

(i) In this case we evaluate $f(x)$ at $x = -2, -1$.
$f(-2) = e^6 - 14 \simeq 389$ and $f(-1) = e^3 - 7 \simeq 13$.
Hence the global maximum of f is $e^6 - 14$ attained at $x = -2$ and the global minimum of f is $e^3 - 7$ attained at $x = -1$.

(ii) In this case we evaluate $f(x)$ at $x = -1, -\frac{1}{3}\ln\left(\frac{7}{3}\right), 0$.
$f(-1) = e^3 - 7 \simeq 13$,
$f\left(-\frac{1}{3}\ln\left(\frac{7}{3}\right)\right) = \exp\left(\ln\left(\frac{7}{3}\right)\right) - \frac{7}{3}\ln\left(\frac{7}{3}\right) = \frac{7}{3} - \frac{7}{3}\ln\left(\frac{7}{3}\right) \simeq 0.4$
and $f(0) = e^0 - 0 = 1$.
Hence the global maximum of f is $e^3 - 7$ attained at $x = -1$ and the global minimum of f is $\frac{7}{3} - \frac{7}{3}\ln\left(\frac{7}{3}\right)$ attained at $x = -\frac{1}{3}\ln\left(\frac{7}{3}\right)$.

(iii) In this case we evaluate $f(x)$ at $x = 0, 1$.
$f(0) = e^0 + 0 = 1$ and $f(1) = e^{-3} + 7 \simeq 7$.
Hence the global maximum of f is $e^{-3} + 7$ attained at $x = 1$ and the global minimum of f is 1 attained at $x = 0$.

(f) Note that we saw in Question 6(f) that the function

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto e^{4x} + 5x \end{aligned}$$

has no critical points.

(i) In this case we evaluate $f(x)$ at $x = -2, -1$.
$f(-2) = e^{-8} - 10 \simeq -10$ and $f(-1) = e^{-4} - 5 \simeq -5$.
Hence the global maximum of f is $e^{-4} - 5$ attained at $x = -1$ and the global minimum of f is $e^{-8} - 10$ attained at $x = -2$.

(ii) In this case we evaluate $f(x)$ at $x = -1, 0$.
$f(-1) = e^{-4} - 5 \simeq -5$ and $f(0) = e^0 - 0 = 1$.
Hence the global maximum of f is 1 attained at $x = 0$ and the global minimum of f is $e^{-4} - 5$ attained at $x = -1$.

(iii) In this case we evaluate $f(x)$ at $x = 0, 1$.
$f(0) = e^0 - 0 = 1$ and $f(1) = e^4 + 5 \simeq 60$.
Hence the global maximum of f is $e^4 + 5$ attained at $x = 1$ and the global minimum of f is 1 attained at $x = 0$.

(g) Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sin(x) \end{aligned}$$

have already been found in Question 6(e): they are $x = \frac{\pi}{2} + k\pi$ for each $k \in \mathbb{Z}$.

(i) Since $\sin\left(\frac{\pi}{2} + 2k\pi\right) = 1$ and $\sin\left(\frac{3\pi}{2} + 2k\pi\right) = -1$ for each $k \in \mathbb{Z}$, the global maximum of f is 1 attained at $x = \frac{\pi}{2} + 2k\pi$ for each $k \in \mathbb{Z}$, and the global minimum of f is -1 attained at $x = \frac{3\pi}{2} + 2k\pi$ for each $k \in \mathbb{Z}$.

(ii) In this case we evaluate $f(x)$ at $x = -\frac{\pi}{4}, \frac{\pi}{4}$. $f\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

Hence the global maximum of f is $\frac{1}{\sqrt{2}}$ attained at $x = \frac{\pi}{4}$ and the global minimum of f is $-\frac{1}{\sqrt{2}}$ attained at $x = -\frac{\pi}{4}$.

(iii) In this case we evaluate $f(x)$ at $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$. $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{3\pi}{2}\right) = -1$ and $f(2\pi) = 0$.

Hence the global maximum of f is 1 attained at $x = \frac{\pi}{2}$ and the global minimum of f is -1 attained at $x = \frac{3\pi}{2}$.

(h) Note that the critical points of

$$\begin{aligned} f\colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \cos(2x) \end{aligned}$$

have already been found in Question 6(f): they are $x = \frac{k\pi}{2}$ for each $k \in \mathbb{Z}$.

(i) Since $\cos(2(k\pi)) = 1$ and $\cos\left(2\left(\frac{\pi}{2} + k\pi\right)\right) = \cos(\pi + 2k\pi) = -1$ for each $k \in \mathbb{Z}$, the global maximum of f is 1 attained at $x = k\pi$ for each $k \in \mathbb{Z}$, and the global minimum of f is -1 attained at $x = \frac{\pi}{2} + k\pi$ for each $k \in \mathbb{Z}$.

(ii) In this case we evaluate $f(x)$ at $x = -\frac{\pi}{4}, 0, \frac{\pi}{4}$.
$f\left(-\frac{\pi}{4}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$, $f(0) = \cos(0) = 1$ and
$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}\right) = 0$.
Hence the global maximum of f is 1 attained at $x = 0$ and the global minimum of f is 0 attained at $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$.

(iii) In this case we evaluate $f(x)$ at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. $f(0) = \cos(0) = 1$, $f\left(\frac{\pi}{2}\right) = \cos(\pi) = -1$, $f(\pi) = \cos(2\pi) = 1$, $f\left(\frac{3\pi}{2}\right) = \cos(3\pi) = -1$, and $f(2\pi) = \cos(4\pi) = 1$.
Hence the global maximum of f is 1 attained at $x = 0$, $x = \pi$ and $x = 2\pi$, and the global minimum of f is -1 attained at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.
The global maxima occur at $(0, 1)$, $(\pi, 1)$ and $(2\pi, 1)$ and the global minima occur at $\left(\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$.

8. Note that we have found all the derivatives and critical points of these functions in Question 6.

   (a) Since $f'(x) = 3x^2 - 6x - 9$, $f''(x) = 6x - 6$.
   We now evaluate $f''(x)$ at each of the critical points.
   $f''(-1) = 6(-1) - 6 = -12 < 0$, so the critical point at $x = -1$ is a local maximum.
   $f''(3) = 6(3) - 6 = 12 > 0$, so the critical point at $x = 3$ is a local minimum.

   (b) Since $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$.
   We now evaluate $f''(x)$ at each of the critical points.
   $f''(-2) = 6(-2) + 6 = -6 < 0$, so the critical point at $x = -2$ is a local maximum.
   $f''(0) = 6(0) + 6 = 6 > 0$, so the critical point at $x = 0$ is a local minimum.

   (c) Since $f'(x) = -6x^2 - 18x + 24$, $f''(x) = -12x - 18$.
   We now evaluate $f''(x)$ at each of the critical points.
   $f''(-4) = -12(-4) - 18 = 30 > 0$, so the critical point at $x = -4$ is a local minimum.
   $f''(1) = -12(1) - 18 = -30 < 0$, so the critical point at $x = 1$ is a local maximum.

   (d) The function $f(x) = 2x^3 + 3x^2 + 6x + 5$ has no critical points to classify.

   (e) Since $f'(x) = -3e^{-3x} + 7$, $f''(x) = 9e^{-3x}$.
   We now evaluate $f''(x)$ at the critical point $x = -\dfrac{1}{3}\ln\left(\dfrac{7}{3}\right)$.
   $f''\left(-\dfrac{1}{3}\ln\left(\dfrac{7}{3}\right)\right) = 9\exp\left(\ln\left(\dfrac{7}{3}\right)\right) = 9\left(\dfrac{7}{3}\right) = 21.$
   Since $f''\left(-\dfrac{1}{3}\ln\left(\dfrac{7}{3}\right)\right) > 0$, f has a local minimum at $x = -\dfrac{1}{3}\ln\left(\dfrac{7}{3}\right)$.

   (f) The function $f(x) = e^{4x} + 5x$ has no critical points to classify.

   (g) Since $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$.
   We now evaluate $f''(x)$ at each of the critical points.
   $f''\left(\dfrac{\pi}{2} + 2k\pi\right) = -\sin\left(\dfrac{\pi}{2} + 2k\pi\right) = -1 < 0$, so the critical points at $x = \dfrac{\pi}{2} + 2k\pi$ are local maxima for each $k \in \mathbb{Z}$.
   $f''\left(\dfrac{3\pi}{2} + 2k\pi\right) = -\sin\left(\dfrac{3\pi}{2} + 2k\pi\right) = -(-1) = 1 > 0$, so the critical points at $x = \dfrac{3\pi}{2} + 2k\pi$ are local minima for each $k \in \mathbb{Z}$.

   (h) Since $f'(x) = -2\sin(2x)$, $f''(x) = -4\cos(2x)$.
   We now evaluate $f''(x)$ at each of the critical points.
   $f''(k\pi) = -4\cos(2k\pi) = -4(1) = -4 < 0$, so the critical points at $x = k\pi$ are local maxima for each $k \in \mathbb{Z}$.
   $f''\left(\dfrac{\pi}{2} + k\pi\right) = -4\cos(\pi + 2k\pi) = -4(-1) = 4 > 0$, so the critical points at $x = \dfrac{\pi}{2} + k\pi$ are local minima for each $k \in \mathbb{Z}$.

9. In this question we will use the iteration formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

(a) In this case $f'(x) = 3x^2 - 6x - 9$, so that $x_{n+1} = x_n - \dfrac{x_n^3 - 3x_n^2 - 9x_n + 12}{3x_n^2 - 6x_n - 9}$.
Since $x_0 = 4$,

$$x_1 = x_0 - \frac{x_0^3 - 3x_0^2 - 9x_0 + 12}{3x_0^2 - 6x_0 - 9} = 4 - \frac{4^3 - 3(4^2) - 9(4) + 12}{3(4^2) - 6(4) - 9} = \frac{68}{15}.$$

Then

$$\begin{aligned} x_2 &= x_1 - \frac{x_1^3 - 3x_1^2 - 9x_1 + 12}{3x_1^2 - 6x_1 - 9} \\ &= \frac{68}{15} - \frac{\left(\frac{68}{15}\right)^3 - 3\left(\frac{68}{15}\right)^2 - 9\left(\frac{68}{15}\right) + 12}{3\left(\frac{68}{15}\right)^2 - 6\left(\frac{68}{15}\right) - 9} \\ &= 4.4268 \text{ to 4 d.p.} \end{aligned}$$

(b) In this case $f'(x) = 3x^2 + 6x$, so that $x_{n+1} = x_n - \dfrac{x_n^3 + 3x_n^2 - 5}{3x_n^2 + 6x_n}$.
Since $x_0 = 1$,

$$x_1 = x_0 - \frac{x_0^3 + 3x_0^2 - 5}{3x_0^2 + 6x_0} = 1 - \frac{1^3 + 3(1^2) - 5}{3(1^2) + 6(1)} = \frac{10}{9}.$$

Then

$$x_2 = x_1 - \frac{x_1^3 + 3x_1^2 - 5}{3x_1^2 + 6x_1} = \frac{10}{9} - \frac{\left(\frac{10}{9}\right)^3 + 3\left(\frac{10}{9}\right)^2 - 5}{3\left(\frac{10}{9}\right)^2 + 6\left(\frac{10}{9}\right)} = 1.1038 \text{ to 4 d.p.}$$

(c) Here $f'(x) = -6x^2 - 18x + 24$, so that $x_{n+1} = x_n - \dfrac{-2x_n^3 - 9x_n^2 + 24x_n - 12}{-6x_n^2 - 18x_n + 24}$.
Since $x_0 = 2$,

$$x_1 = x_0 - \frac{-2x_0^3 - 9x_0^2 + 24x_0 - 12}{-6x_0^2 - 18x_0 + 24} = 2 - \frac{-2(2^3) - 9(2^2) + 24(2) - 12}{-6(2^2) - 18(2) + 24} = \frac{14}{9}.$$

Then

$$\begin{aligned} x_2 &= x_1 - \frac{-2x_1^3 - 9x_1^2 + 24x_1 - 12}{-6x_1^2 - 18x_1 + 24} \\ &= \frac{14}{9} - \frac{-2\left(\frac{14}{9}\right)^3 - 9\left(\frac{14}{9}\right)^2 + 24\left(\frac{14}{9}\right) - 12}{-6\left(\frac{14}{9}\right)^2 - 18\left(\frac{14}{9}\right) + 24} \\ &= 1.3410 \text{ to 4 d.p.} \end{aligned}$$

(d) Here $f'(x) = -\sin(x) - 1$, so that $x_{n+1} = x_n - \dfrac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$.
Since $x_0 = 0$,

$$x_1 = x_0 - \frac{\cos(x_0) - x_0}{-\sin(x_0) - 1} = 0 - \frac{\cos(0) - 0}{-\sin(0) - 1} = 1.$$

Then

$$x_2 = x_1 - \frac{\cos(x_1) - x_1}{-\sin(x_1) - 1} = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} = 0.7504 \text{ to 4 d.p.}$$