Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Chapter 4 Solutions

In all these solutions, c will represent an arbitrary constant.

1. (a) Since $f(x) = 5$ is a constant, $\displaystyle\int_0^1 5\,dx = [5x]_0^1 = 5$.

   (b) Since $f(x) = -\pi\cos(e)$ is a constant, $\displaystyle\int -\pi\cos(e)\,dx = -\pi\cos(e)x + c$.

   (c) Since $f(x) = x^2$ is of the form $f(x) = x^n$ with $n = 2$,

   $$\int_{-1}^{1} x^2\,dx = \left[\frac{x^{2+1}}{2+1}\right]_{-1}^{1} = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

   (d) Since $f(x) = x^{\frac{9}{2}}$ is of the form $f(x) = x^n$ with $n = \dfrac{9}{2}$,

   $$\int x^{\frac{9}{2}}\,dx = \frac{1}{9/2+1}x^{\frac{9}{2}+1} + c = \frac{2}{11}x^{\frac{11}{2}} + c.$$

   (e) Since $f(x) = x^{-5}$ is of the form $f(x) = x^n$ with $n = -5$,

   $$\begin{aligned}\int_1^2 x^{-5}\,dx &= \left[\frac{1}{-5+1}x^{-5+1}\right]_1^2\\ &= \left[-\frac{1}{4}x^{-4}\right]_1^2\\ &= -\frac{1}{4}2^{-4} - \left(-\frac{1}{4}1^{-4}\right)\\ &= -\frac{1}{64} + \frac{1}{4}\\ &= \frac{15}{64}.\end{aligned}$$

   (f) Since $f(x) = x^{\cos(2)}$ is of the form $f(x) = x^n$ with $n = \cos(2)$,

   $$\int x^{\cos(2)}\,dx = \frac{1}{\cos(2)+1}x^{\cos(2)+1} + c.$$

   (g) Since $f(x) = e^{4x}$ is of the form $f(x) = e^{ax}$ with $a = 4$,

   $$\int_0^2 e^{4x}\,dx = \left[\frac{1}{4}e^{4x}\right]_0^2 = \frac{1}{4}e^{4(2)} - \frac{1}{4}e^0 = \frac{1}{4}(e^8 - 1).$$

(h) Since $f(x) = e^{\frac{3}{2}x}$ is of the form $f(x) = e^{ax}$ with $a = \frac{3}{2}$,

$$\int e^{\frac{3}{2}x}\,dx = \frac{1}{3/2}e^{\frac{3}{2}x} + c = \frac{2}{3}e^{\frac{3}{2}x} + c.$$

(i) Since $f(x) = e^{-6x}$ is of the form $f(x) = e^{ax}$ with $a = -6$,

$$\begin{aligned}\int_{-1}^{0} e^{-6x}\,dx &= \left[\frac{1}{-6}e^{-6x}\right]_{-1}^{0}\\ &= \left[-\frac{1}{6}e^{-6x}\right]_{-1}^{0}\\ &= -\frac{1}{6}e^{0} - \left(-\frac{1}{6}e^{-6(-1)}\right)\\ &= \frac{1}{6}\left(e^{6} - 1\right).\end{aligned}$$

(j) Since $f(x) = e^{\pi x}$ is of the form $f(x) = e^{ax}$ with $a = \pi$, $\displaystyle\int e^{\pi x}\,dx = \frac{1}{\pi}e^{\pi x} + c.$

(k) $\displaystyle\int_{1}^{2} \frac{1}{x}\,dx = [\ln(x)]_{1}^{2} = \ln(2) - \ln(1) = \ln(2) - 0 = \ln(2).$

(l) Since $f(x) = \sin(2x)$ is of the form $f(x) = \sin(ax)$ with $a = 2$,

$$\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + c.$$

(m) Since $f(x) = \sin(-3x)$ is of the form $f(x) = \sin(ax)$ with $a = -3$,

$$\begin{aligned}\int_{0}^{\frac{\pi}{3}} \sin(-3x)\,dx &= \left[-\frac{1}{-3}\cos(-3x)\right]_{0}^{\frac{\pi}{3}}\\ &= \left[\frac{1}{3}\cos(-3x)\right]_{0}^{\frac{\pi}{3}}\\ &= \frac{1}{3}\cos\left(-3\cdot\frac{\pi}{3}\right) - \frac{1}{3}\cos(-3(0))\\ &= \frac{1}{3}\cos(-\pi) - \frac{1}{3}\cos(0)\\ &= \frac{1}{3}(-1) - \frac{1}{3}(1)\\ &= -\frac{2}{3}.\end{aligned}$$

(n) Since $f(x) = \sin(ex)$ is of the form $f(x) = \sin(ax)$ with $a = e$,

$$\int \sin(ex)\,dx = -\frac{1}{e}\cos(ex) + c.$$

(o) Since $f(x) = \cos(3x)$ is of the form $f(x) = \cos(ax)$ with $a = 3$,

$$\begin{aligned}
\int_{-\pi}^{\frac{\pi}{3}} \cos(3x)\,dx &= \left[\frac{1}{3}\sin(3x)\right]_{-\pi}^{\frac{\pi}{3}} \\
&= \frac{1}{3}\sin\left(3\cdot\frac{\pi}{3}\right) - \frac{1}{3}\sin(3(-\pi)) \\
&= \frac{1}{3}\sin(\pi) - \frac{1}{3}\sin(-3\pi) \\
&= \frac{1}{3}(0) - \frac{1}{3}(0) \\
&= 0.
\end{aligned}$$

(p) Since $f(x) = \cos(-\pi x)$ is of the form $f(x) = \cos(ax)$ with $a = -\pi$,

$$\int \cos(-\pi x)\,dx = \frac{1}{-\pi}\sin(-\pi x) + c = -\frac{1}{\pi}\sin(-\pi x) + c.$$

2. (a) We will first use the sum and multiple rules to find the corresponding definite integral.

$$\begin{aligned}
&\int 1 + 3x - 2x^2 + 3x^3 - 4x^4\,dx \\
&\quad= \int 1\,dx + \int 3x\,dx + \int -2x^2\,dx + \int 3x^3\,dx + \int -4x^4\,dx \\
&\quad= \int 1\,dx + 3\int x\,dx - 2\int x^2\,dx + 3\int x^3\,dx - 4\int x^4\,dx \\
&\quad= x + 3\left(\frac{1}{2}x^2\right) - 2\left(\frac{1}{3}x^3\right) + 3\left(\frac{1}{4}x^4\right) - 4\left(\frac{1}{5}x^5\right) + c \\
&\quad= x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{4}{5}x^5 + c.
\end{aligned}$$

Note that in your assignment or exam solutions you don't need to give as much detail as this. I am just setting out everything carefully until you get used to the ideas involved.
Hence

$$\begin{aligned}
\int_{-1}^{1} 1 + 3x - 2x^2 + 3x^3 - 4x^4\,dx &= \left[x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{4}{5}x^5\right]_{-1}^{1} \\
&= 1 + \frac{3}{2}(1^2) - \frac{2}{3}(1^3) + \frac{3}{4}(1^4) - \frac{4}{5}(1^5) \\
&\quad - \left[-1 + \frac{3}{2}(-1)^2 - \frac{2}{3}(-1)^3 + \frac{3}{4}(-1)^4 - \frac{4}{5}(-1)^5\right] \\
&= \frac{107}{60} - \frac{163}{60} \\
&= -\frac{14}{15}.
\end{aligned}$$

(b) Using the sum and multiple rules,

$$\begin{aligned}\int -x^{-1} + 2\sin 4x\, dx &= \int -x^{-1}\, dx + \int 2\sin 4x\, dx \\ &= -\int x^{-1}\, dx + 2\int \sin 4x\, dx \\ &= -\int \frac{1}{x}\, dx + 2\int \sin 4x\, dx \\ &= -\ln(x) + 2\left(-\frac{1}{4}\cos(4x)\right) + c \\ &= -\ln(x) - \frac{1}{2}\cos(4x) + c.\end{aligned}$$

(c) We will first use the sum and multiple rules to find the corresponding definite integral.

$$\begin{aligned}\int 3e^{-\frac{1}{2}x} - 2\cos\left(\frac{1}{2}x\right)\, dx &= \int 3e^{-\frac{1}{2}x}\, dx + \int -2\cos\left(\frac{1}{2}x\right)\, dx \\ &= 3\int e^{-\frac{1}{2}x}\, dx - 2\int \cos\left(\frac{1}{2}x\right)\, dx \\ &= 3\left(\frac{1}{-1/2}e^{-\frac{1}{2}x}\right) - 2\left(\frac{1}{1/2}\sin\left(\frac{1}{2}x\right)\right) + c \\ &= -6e^{-\frac{1}{2}x} - 4\sin\left(\frac{1}{2}x\right).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi} 3e^{-\frac{1}{2}x} - 2\cos\left(\frac{1}{2}x\right)\, dx &= \left[-6e^{-\frac{1}{2}x} - 4\sin\left(\frac{1}{2}x\right)\right]_0^{\pi} \\ &= -6e^{-\frac{1}{2}\pi} - 4\sin\left(\frac{1}{2}\pi\right) - \left[-6e^{-\frac{1}{2}(0)} - 4\sin\left(\frac{1}{2}(0)\right)\right] \\ &= -6e^{-\frac{\pi}{2}} - 4(1) - [-6(1) - 4(0)] \\ &= 2 - 6e^{-\frac{\pi}{2}}.\end{aligned}$$

(d) Using the sum and multiple rules,

$$\begin{aligned}\int 4\cos(-3x) - e^{-\frac{3}{2}x}\, dx &= \int 4\cos(-3x)\, dx + \int -e^{-\frac{3}{2}x}\, dx \\ &= 4\int \cos(-3x)\, dx - \int e^{-\frac{3}{2}x}\, dx \\ &= 4\left(\frac{1}{-3}\sin(-3x)\right) - \left(\frac{1}{-3/2}e^{-\frac{3}{2}x}\right) + c \\ &= -\frac{4}{3}\sin(-3x) + \frac{2}{3}e^{-\frac{3}{2}x} + c.\end{aligned}$$

(e) We will first use the sum and multiple rules to find the corresponding definite integral.

$$\begin{aligned}\int -2x^2 + e^{\cos(1)x}\,dx &= \int -2x^2\,dx + \int e^{\cos(1)x}\,dx\\ &= -2\int x^2\,dx + \int e^{\cos(1)x}\,dx\\ &= -2\left(\frac{1}{3}x^3\right) + \frac{1}{\cos(1)}e^{\cos(1)x} + c\\ &= -\frac{2}{3}x^3 + \frac{1}{\cos(1)}e^{\cos(1)x} + c.\end{aligned}$$

Hence

$$\begin{aligned}\int_{-2}^{1} -2x^2 + e^{\cos(1)x}\,dx &= \left[-\frac{2}{3}x^3 + \frac{1}{\cos(1)}e^{\cos(1)x}\right]_{-2}^{1}\\ &= -\frac{2}{3}(1^3) + \frac{1}{\cos(1)}e^{\cos(1)(1)} - \left[-\frac{2}{3}(-2)^3 + \frac{1}{\cos(1)}e^{\cos(1)(-2)}\right]\\ &= \frac{1}{\cos(1)}\left(e^{\cos(1)} - e^{-2\cos(1)}\right) - 6.\end{aligned}$$

(f) Using the sum and multiple rules,

$$\begin{aligned}&\int 2\sin(3x) - 3\sin(2x) + 2\cos(3x) - 3\cos(2x)\,dx\\ &= \int 2\sin(3x)\,dx + \int -3\sin(2x)\,dx + \int 2\cos(3x)\,dx + \int -3\cos(2x)\,dx\\ &= 2\int \sin(3x)\,dx - 3\int \sin(2x)\,dx + 2\int \cos(3x)\,dx - 3\int \cos(2x)\,dx\\ &= 2\left(-\frac{1}{3}\cos(3x)\right) - 3\left(-\frac{1}{2}\cos(2x)\right) + 2\left(\frac{1}{3}\sin(3x)\right) - 3\left(\frac{1}{2}\sin(2x)\right) + c\\ &= -\frac{2}{3}\cos(3x) + \frac{3}{2}\cos(2x) + \frac{2}{3}\sin(3x) - \frac{3}{2}\sin(2x) + c\end{aligned}$$

(g) We will first use the sum rule to find the corresponding definite integral.

$$\begin{aligned}\int e^2 + e^{2x} - 4\,dx &= \int e^2 - 4\,dx + \int e^{2x}\,dx\\ &= (e^2 - 4)x + \frac{1}{2}e^{2x} + c.\end{aligned}$$

Note that we didn't need the multiple rule here and also note that we could deal with $e^2 - 4$ all at once since $e^2 - 4$ is a constant.
Hence

$$\begin{aligned}\int_{1}^{3} e^2 + e^{2x} - 4\,dx &= \left[(e^2 - 4)x + \frac{1}{2}e^{2x}\right]_{1}^{3}\\ &= (e^2 - 4)(3) + \frac{1}{2}e^{2(3)} - \left[(e^2 - 4)(1) + \frac{1}{2}e^{2(1)}\right]\\ &= 2(e^2 - 4) + \frac{1}{2}\left(e^6 - e^2\right)\\ &= \frac{1}{2}e^6 + \frac{3}{2}e^2 - 8.\end{aligned}$$

(h) Using the sum and multiple rules,

$$\begin{aligned}
\int -3x^{-3} + 4x^4 + 5x^{-5} + 3x^0\,dx \\
&= \int -3x^{-3}\,dx + \int 4x^4\,dx + \int 5x^{-5}\,dx + \int 3x^0\,dx \\
&= -3\int x^{-3}\,dx + 4\int x^4\,dx + 5\int x^{-5}\,dx + \int 3\,dx \\
&= -3\left(\frac{1}{-3+1}x^{-3+1}\right) + 4\left(\frac{1}{4+1}x^{4+1}\right) + 5\left(\frac{1}{-5+1}x^{-5+1}\right) + 3x + c \\
&= \frac{3}{2}x^{-2} + \frac{4}{5}x^5 - \frac{5}{4}x^{-4} + 3x + c.
\end{aligned}$$

For the remainder of the questions we will still need the sum and multiple rules but I will not mention them explicitly.

3. (a) Here we use integration by substitution.
Let $u = x + 5$, so that $\frac{du}{dx} = 1$.
Then $dx = \frac{du}{du/dx} = \frac{du}{1} = du$.
Also, when $x = 0$, $u = 5$ and when $x = 1$, $u = 6$.
Hence

$$\int_0^1 (x+5)^9\,dx = \int_5^6 u^9\,du = \left[\frac{1}{10}u^{10}\right]_5^6 = \frac{1}{10}6^{10} - \frac{1}{10}5^{10} = \frac{1}{10}(6^{10} - 5^{10}).$$

(b) Here we use integration by substitution.
Let $u = x^3 - 5$, so that $\frac{du}{dx} = 3x^2$.
Then $dx = \frac{du}{du/dx} = \frac{du}{3x^2}$.
Hence

$$\begin{aligned}
\int x^2(x^3-5)^5\,dx &= \int x^2u^5 \cdot \frac{du}{3x^2} \\
&= \int \frac{1}{3}u^5\,du \\
&= \frac{1}{3}\left(\frac{1}{6}u^6\right) + c \\
&= \frac{1}{18}u^6 + c \\
&= \frac{1}{18}\left(x^3-5\right)^6 + c.
\end{aligned}$$

(c) Here we use integration by substitution.
Let $u = \cos(x)$, so that $\frac{du}{dx} = -\sin(x)$. Note we chose $u = \cos(x)$ rather than $u = \sin(x)$ since this ensures the $\sin(x)$ in the numerator cancels. This is part of the skill you will have to learn when performing integration; what

works and what doesn't.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{-\sin(x)}$.
Also, when $x = 0$, $u = 1$ and when $x = \dfrac{\pi}{4}$, $u = \dfrac{1}{\sqrt{2}}$.
Hence

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}\, dx &= \int_1^{\frac{1}{\sqrt{2}}} \frac{\sin(x)}{u} \cdot \frac{du}{-\sin(x)} \\
&= \int_1^{\frac{1}{\sqrt{2}}} -\frac{1}{u}\, du \\
&= [-\ln(u)]_1^{\frac{1}{\sqrt{2}}} \\
&= -\ln\left(\frac{1}{\sqrt{2}}\right) - (-\ln(1)) \\
&= -\ln\left(2^{-\frac{1}{2}}\right) + 0 \\
&= -\left(-\frac{1}{2}\ln(2)\right) \\
&= \frac{1}{2}\ln(2).
\end{aligned}$$

(d) Here we use integration by substitution.
Let $u = \sin(x)$, so that $\dfrac{du}{dx} = \cos(x)$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{\cos(x)}$.
Hence

$$\int \frac{\cos(x)}{\sin(x)}\, dx = \int \frac{\cos(x)}{u} \cdot \frac{du}{\cos(x)} = \int \frac{1}{u}\, du = \ln(u) + c = \ln(\sin(x)) + c.$$

(e) Here we use integration by substitution.
Let $u = \dfrac{1}{x}$, so that $\dfrac{du}{dx} = -\dfrac{1}{x^2}$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{-1/x^2} = -x^2 du$.
Also, when $x = 1$, $u = 1$ and when $x = 2$, $u = \dfrac{1}{2}$.
Hence

$$\begin{aligned}
\int_1^2 x^{-2} e^{\frac{1}{x}}\, dx &= \int_1^{\frac{1}{2}} \frac{1}{x^2} e^u (-x^2)\, du \\
&= \int_1^{\frac{1}{2}} -e^u\, du \\
&= [-e^u]_1^{\frac{1}{2}} \\
&= -e^{\frac{1}{2}} - (-e^1) \\
&= e - e^{\frac{1}{2}} \\
&= e - \sqrt{e}.
\end{aligned}$$

(f) Here we use integration by substitution.
Let $u = x^4 - x^3 + x^2 - x$, so that $\dfrac{du}{dx} = 4x^3 - 3x^2 + 2x - 1$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{4x^3 - 3x^2 + 2x - 1}$.
Hence

$$\begin{aligned}
\int \frac{12x^3 - 9x^2 + 6x - 3}{x^4 - x^3 + x^2 - x}\, dx &= \int \frac{12x^3 - 9x^2 + 6x - 3}{u} \cdot \frac{du}{4x^3 - 3x^2 + 2x - 1} \\
&= \int 3 \cdot \frac{1}{u}\, du \\
&= 3\ln(u) + c \\
&= 3\ln(x^4 - x^3 + x^2 - x) + c.
\end{aligned}$$

(g) Here we use integration by substitution.
Let $u = 1 - x^2$, so that $\dfrac{du}{dx} = -2x$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{-2x}$.
Also, when $x = 0$, $u = 1$ and when $x = 1$, $u = 0$.
Hence

$$\begin{aligned}
\int_0^1 x\sqrt{1 - x^2}\, dx &= \int_1^0 x\sqrt{u} \cdot \frac{du}{-2x} \\
&= \int_1^0 -\frac{1}{2} u^{\frac{1}{2}}\, du \\
&= \left[-\frac{1}{2} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \right]_1^0 \\
&= \left[-\frac{1}{3} u^{\frac{3}{2}} \right]_1^0 \\
&= -\frac{1}{3} 0^{\frac{3}{2}} - \left(-\frac{1}{3} 1^{\frac{3}{2}} \right) \\
&= 0 + \frac{1}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

(h) Here we use integration by substitution.
Let $u = x^4 + \dfrac{4}{3}x^3$, so that $\dfrac{du}{dx} = 4x^3 + 4x^2$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{4x^3 + 4x^2}$.

Hence

$$\begin{aligned}
\int (x^3 + x^2) \sin\left(x^4 + \frac{4}{3}x^3\right) dx &= \int (x^3 + x^2) \sin(u) \cdot \frac{du}{4x^3 + 4x^2} \\
&= \int \frac{1}{4} \sin(u)\, du \\
&= \frac{1}{4}(-\cos(u)) + c \\
&= -\frac{1}{4} \cos\left(x^4 + \frac{4}{3}x^3\right) + c.
\end{aligned}$$

(i) Here we use integration by substitution.
Let $u = x^2 + 4$, so that $\dfrac{du}{dx} = 2x$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{2x}$.
Also, when $x = 0$, $u = 4$ and when $x = 1$, $u = 5$.
Hence

$$\begin{aligned}
\int_0^1 \frac{x}{\sqrt{x^2 + 4}}\, dx &= \int_4^5 \frac{x}{\sqrt{u}} \cdot \frac{du}{2x} \\
&= \int_4^5 \frac{1}{2} u^{-\frac{1}{2}}\, du \\
&= \left[\frac{1}{2}\left(2u^{\frac{1}{2}}\right)\right]_4^5 \\
&= \left[u^{\frac{1}{2}}\right]_4^5 \\
&= 5^{\frac{1}{2}} - 4^{\frac{1}{2}} \\
&= \sqrt{5} - 2.
\end{aligned}$$

(j) Here we use integration by substitution.
Let $u = x^3 + 1$, so that $\dfrac{du}{dx} = 3x^2$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{3x^2}$.
Hence

$$\begin{aligned}
\int x^2 \left(x^3 + 1\right)^{\frac{3}{2}}\, dx &= \int x^2 u^{\frac{3}{2}} \cdot \frac{du}{3x^2} \\
&= \int \frac{1}{3} u^{\frac{3}{2}}\, du \\
&= \frac{1}{3}\left(\frac{2}{5} u^{\frac{5}{2}}\right) + c \\
&= \frac{2}{15} u^{\frac{5}{2}} + c \\
&= \frac{2}{15}\left(x^3 + 1\right)^{\frac{5}{2}}.
\end{aligned}$$

(k) Here we use integration by substitution.
Let $u = \cos(x)$, so that $\dfrac{du}{dx} = -\sin(x)$.

Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{-\sin(x)}$.
Also, when $x = 0$, $u = 1$ and when $x = \dfrac{\pi}{2}$, $u = 0$.
Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(x)e^{\cos(x)}\,dx &= \int_1^0 \sin(x)e^u \cdot \frac{du}{-\sin(x)} \\ &= \int_1^0 -e^u\,du \\ &= \left[-e^u\right]_1^0 \\ &= -e^0 - (-e^1) \\ &= -1 + e \\ &= e - 1.\end{aligned}$$

(l) Here we use integration by substitution.
Let $u = x^4$, so that $\dfrac{du}{dx} = 4x^3$.
Then $dx = \dfrac{du}{du/dx} = \dfrac{du}{4x^3}$.
Hence

$$\begin{aligned}\int x^3\cos\left(x^4\right)\,dx &= \int x^3\cos(u)\cdot\frac{du}{4x^3} \\ &= \int \frac{1}{4}\cos(u)\,du \\ &= \frac{1}{4}\sin(u) + c \\ &= \frac{1}{4}\sin\left(x^4\right) + c.\end{aligned}$$

(m) Here we use integration by substitution.
Let $x = \sin(u)$, so that $\dfrac{dx}{du} = \cos(u)$.
Then $dx = \dfrac{dx}{du}du = \cos(u)du$.
Also, when $x = 0$, $u = 0$ and when $x = 1$, $u = \dfrac{\pi}{2}$.
Hence

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,dx &= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2(u)}\cos(u)\,du \\ &= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2(u)}\cos(u)\,du \\ &= \int_0^{\frac{\pi}{2}} \cos(u)\cdot\cos(u)\,du \\ &= \int_0^{\frac{\pi}{2}} \cos^2(u)\,du \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2u)\,du\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{2}u + \frac{1}{2}\left(\frac{1}{2}\sin(2u)\right)\right]_0^{\frac{\pi}{2}} \\
&= \left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right]_0^{\frac{\pi}{2}} \\
&= \frac{\pi}{4} + \frac{1}{4}\sin(\pi) - \left(0 + \frac{1}{4}\sin(0)\right) \\
&= \frac{\pi}{4} + 0 - (0+0) \\
&= \frac{\pi}{4}.
\end{aligned}
$$

We don't cover this in MATH00040 but the equation of a circle of radius one centred at the origin is $x^2 + y^2 = 1$ and on solving this for y we obtain $y = \pm\sqrt{1-x^2}$. So what we have actually calculated here is the area inside this circle that lies in the first quadrant. The area inside a circle of radius $r = 1$ is $\pi r^2 = \pi 1^2 = \pi$ and we have found a quarter of this area, that is $\frac{\pi}{4}$. Note in addition that if we replace $\sqrt{1-x^2}$ with $\sqrt{r^2-x^2}$ and the upper limit 1 with r then we can actually *derive* the formula πr^2 for the area inside a circle. See if you can decide what the substitution should be in this case and then derive the formula.

4. (a) Here we use integration by parts.
Let $f(x) = 3x$ and $g'(x) = e^{2x}$, so that $f'(x) = 3$ and $g(x) = \frac{1}{2}e^{2x}$.
Hence, using the integration by parts formula,

$$
\begin{aligned}
\int 3xe^{2x}\,dx &= 3x \cdot \frac{1}{2}e^{2x} - \int 3 \cdot \frac{1}{2}e^{2x}\,dx \\
&= \frac{3}{2}xe^{2x} - \int \frac{3}{2}e^{2x}\,dx \\
&= \frac{3}{2}xe^{2x} - \frac{3}{2}\left(\frac{1}{2}e^{2x}\right) + c \\
&= \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + c
\end{aligned}
$$

(b) Here we use integration by parts.
Let $f(x) = 5x$ and $g'(x) = \cos(2x)$, so that $f'(x) = 5$ and $g(x) = \frac{1}{2}\sin(2x)$.
Hence, using the integration by parts formula,

$$
\begin{aligned}
\int_0^\pi 3x\cos(2x)\,dx &= \left[5x\left(\frac{1}{2}\sin(2x)\right)\right]_0^\pi - \int_0^\pi \frac{5}{2}\sin(2x)\,dx \\
&= \left[\frac{5}{2}x\sin(2x)\right]_0^\pi - \int_0^\pi \frac{5}{2}\sin(2x)\,dx \\
&= \frac{5}{2}(\pi)\sin(2\pi) - \frac{5}{2}(0)\sin(2(0)) - \left[\frac{5}{2}\left(-\frac{1}{2}\cos(2x)\right)\right]_0^\pi \\
&= 0 - 0 + \left[\frac{5}{4}\cos(2x)\right]_0^\pi \\
&= \frac{5}{4}\cos(2\pi) - \frac{5}{4}\cos(2(0)) = \frac{5}{4} - \frac{5}{4} = 0.
\end{aligned}
$$

(c) Here we use integration by parts.

Let $f(x) = \ln(x)$ and $g'(x) = x$, so that $f'(x) = \frac{1}{x}$ and $g(x) = \frac{1}{2}x^2$.

Hence, using the integration by parts formula,

$$\begin{aligned}\int x\ln(x)\,dx &= = \ln(x)\left(\frac{1}{2}x^2\right) - \int \frac{1}{x}\left(\frac{1}{2}x^2\right)\,dx \\ &= \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx \\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + c.\end{aligned}$$

(d) Here we use integration by parts.

Let $f(x) = \ln(x)$ and $g'(x) = x^4 + 3$, so that $f'(x) = \frac{1}{x}$ and $g(x) = \frac{1}{5}x^5 + 3x$.

Hence, using the integration by parts formula,

$$\begin{aligned}\int_1^2 \left(x^4+3\right)\ln(x)\,dx &= \left[\ln(x)\left(\frac{1}{5}x^5 + 3x\right)\right]_1^2 - \int_1^2 \frac{1}{x}\left(\frac{1}{5}x^5+3x\right)\,dx \\ &= \left[\left(\frac{1}{5}x^5+3x\right)\ln(x)\right]_1^2 - \int_1^2 \frac{1}{5}x^4 + 3\,dx \\ &= \left(\frac{1}{5}2^5 + 3(2)\right)\ln(2) - \left(\frac{1}{5}1^5 + 3(1)\right)\ln(1) - \left[\frac{1}{25}x^5 + 3x\right]_1^2 \\ &= \left(\frac{32}{5} + 6\right)\ln(2) - \left(\frac{1}{5} + 3\right)\ln(1) \\ &\quad - \left[\left(\frac{1}{25}2^5 + 3(2)\right) - \left(\frac{1}{25}1^5 + 3(1)\right)\right] \\ &= \frac{62}{5}\ln(2) - 0 - \left[\frac{182}{25} - \frac{76}{25}\right] \\ &= \frac{62}{5}\ln(2) - \frac{106}{25}.\end{aligned}$$

(e) Here we will have to integration by parts twice.

First we let $f(x) = x^2$ and $g'(x) = \sin(x)$, so that $f'(x) = 2x$ and $g(x) = -\cos(x)$.

Hence, using the integration by parts formula,

$$\begin{aligned}\int x^2\sin(x)\,dx &= x^2\left(-\cos(x)\right) - \int 2x\left(-\cos(x)\right)\,dx \\ &= -x^2\cos(x) + \int 2x\cos(x)\,dx. \qquad (1)\end{aligned}$$

We will now find $\int 2x\cos(x)\,dx$ by integrating by parts again.

So let $f(x) = 2x$ and $g'(x) = \cos(x)$, so that $f'(x) = 2$ and $g(x) = \sin(x)$.

Then, using the integration by parts formula,

$$\begin{aligned}\int 2x\cos(x)\,dx &= 2x\sin(x) - \int 2\sin x\,dx \\ &= 2x\sin(x) - 2(-\cos(x)) + c \\ &= 2x\sin(x) + 2\cos(x) + c. \qquad (2)\end{aligned}$$

If we now substitute (2) into (1), we finally obtain

$$\int x^2 \sin(x)\, dx = -x^2 \cos(x) + 2x \sin(x) + 2\cos(x) + c.$$

(f) Again here we will have to integrate by parts twice.
We will first find the associated definite integral $\int x^2 e^x\, dx$.
Let $f(x) = x^2$ and $g'(x) = e^x$, so that $f'(x) = 2x$ and $g(x) = e^x$.
Hence, using the integration by parts formula,

$$\int x^2 e^x\, dx = x^2 e^x - \int 2xe^x\, dx \tag{3}$$

We will now find $\int 2xe^x\, dx$ by integrating by parts again.
Let $f(x) = 2x$ and $g'(x) = e^x$, so that $f'(x) = 2$ and $g(x) = e^x$.
Hence, using the integration by parts formula,

$$\int 2xe^x\, dx = 2xe^x - \int 2e^x\, dx = 2xe^x - 2e^x. \tag{4}$$

If we now substitute (4) into (3), we see that

$$\int x^2 e^x\, dx = x^2 e^x - (2xe^x - 2e^x) + c = x^2 e^x - 2xe^x + 2e^x + c.$$

Hence

$$\begin{aligned}\int_0^1 x^2 e^x\, dx &= \left[x^2 e^x - 2xe^x + 2e^x\right]_0^1 \\ &= 1^2 e^1 - 2(1)e^1 + 2e^1 - [0^2 e^0 - 2(0)e^0 + 2e^0] \\ &= e - 2e + 2e - [0 - 0 + 2] \\ &= e - 2.\end{aligned}$$

5. (a) Here we use partial fractions.
Since $x^2 - 4 = (x-2)(x+2)$, we let

$$\frac{3x+2}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}, \tag{5}$$

where A and B are constants we have to find. Multiplying both sides of (5) by $x^2 - 4$ we obtain

$$3x + 2 = A(x+2) + B(x-2). \tag{6}$$

If we let $x = -2$ in (6), we obtain $-4 = -4B$, so that $B = 1$.
If we let $x = 2$ in (6), we obtain $8 = 4A$, so that $A = 2$.
Hence

$$\begin{aligned}\int \frac{3x+2}{x^2-4}\, dx &= \int \frac{2}{x-2} + \frac{1}{x+2}\, dx \\ &= 2\ln(x-2) + \ln(x+2) + c.\end{aligned}$$

Note that if you can't 'spot' the integrals, then you can use the substitutions $u = x - 2$ and $u = x + 2$ for the two terms, respectively. Similar substitutions will work in all the examples below but I won't explicitly mention them.

(b) Again we use partial fractions.
Since $x^2 + 9x + 20 = (x + 4)(x + 5)$, we let

$$\frac{-1}{x^2 + 9x + 20} = \frac{A}{x + 4} + \frac{B}{x + 5}, \tag{7}$$

where A and B are constants we have to find. Multiplying both sides of (7) by $x^2 + 9x + 20$ we obtain

$$-1 = A(x + 5) + B(x + 4). \tag{8}$$

If we let $x = -5$ in (8), we obtain $-1 = -B$, so that $B = 1$.
If we let $x = -4$ in (8), we obtain $-1 = A$, so that $A = -1$.
Hence

$$\begin{aligned}\int_0^1 \frac{-1}{x^2 + 9x + 20}\,dx &= \int_0^1 \frac{-1}{x + 4} + \frac{1}{x + 5}\,dx \\ &= [-\ln(x + 4) + \ln(x + 5)]_0^1 \\ &= -\ln(5) + \ln(6) - (-\ln(4) + \ln(5)) \\ &= \ln(6) - 2\ln(5) + \ln(4)\end{aligned}$$

(c) Again we use partial fractions.
Since $x^2 + 4x = x(x + 4)$, we let

$$\frac{-x - 8}{x^2 + 4x} = \frac{A}{x} + \frac{B}{x + 4}, \tag{9}$$

where A and B are constants we have to find. Multiplying both sides of (9) by $x^2 + 4x$ we obtain

$$-x - 8 = A(x + 4) + Bx. \tag{10}$$

If we let $x = -4$ in (10), we obtain $-4 = -4B$, so that $B = 1$.
If we let $x = 0$ in (10), we obtain $-8 = 4A$, so that $A = -2$.
Hence

$$\begin{aligned}\int \frac{-x - 8}{x^2 + 4x}\,dxx &= \int \frac{-2}{x} + \frac{1}{x + 4}\,dx \\ &= -2\ln(x) + \ln(x + 4) + c \quad = \ln(x + 4) - 2\ln(x) + c.\end{aligned}$$

(d) Again we use partial fractions.
Since $(x^2 - 1)(x + 2) = (x - 1)(x + 1)(x + 2)$, we let

$$\frac{3x^2 + 4x - 1}{(x^2 - 1)(x + 2)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 2}, \tag{11}$$

where A, B and C are constants we have to find. Multiplying both sides of (11) by $(x^2 - 1)(x + 2)$ we obtain

$$3x^2 + 4x - 1 = A(x + 1)(x + 2) + B(x - 1)(x + 2) + C(x - 1)(x + 1). \tag{12}$$

If we let $x = -2$ in (12), we obtain $3 = 3C$, so that $C = 1$.
If we let $x = -1$ in (12), we obtain $-2 = -2B$, so that $B = 1$.
If we let $x = 1$ in (12), we obtain $6 = 6A$, so that $A = 1$.
Hence

$$\begin{aligned}\int_2^3 \frac{3x^2+4x-1}{(x^2-1)(x+2)}\,dx &= \int_2^3 \frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x+2}\,dx \\ &= [\ln(x-1)+\ln(x+1)+\ln(x+2)]_2^3 \\ &= \ln(2)+\ln(4)+\ln(5)-(\ln(1)+\ln(3)+\ln(4)) \\ &= \ln(2)+\ln(5)-\ln(3).\end{aligned}$$

(e) Again we use partial fractions.
Ths time we have a repeated root, so we let

$$\frac{x-1}{(x-2)^2} = \frac{A}{(x-2)^2}+\frac{B}{x-2}, \tag{13}$$

where A and B are constants we have to find. Multiplying both sides of (13) by $(x-2)^2$ we obtain

$$x-1 = A + B(x-2) \iff x-1 = Bx + (A-2B).$$

Comparing the coefficients of x we obtain $B = 1$ and comparing the constant terms we obtain $-1 = A - 2B$. Since $B = 1$, this means that $A = 1$ as well.
Hence

$$\begin{aligned}\int \frac{x-1}{(x-2)^2}\,dx &= \int \frac{1}{(x-2)^2}+\frac{1}{x-2}\,dx \\ &= -\frac{1}{x-2}+\ln(x-2)+c \\ &= \ln(x-2)-\frac{1}{x-2}+c.\end{aligned}$$

Note the substitution $u = x-2$ will also work for the $\frac{1}{(x-2)^2}$ term here.

(f) Again we use partial fractions.
Ths time we have a quadratic term in the denominator which doesn't factorise, so we let

$$\frac{3x^2+6x+4}{(x^2+2x+2)(x+1)} = \frac{Ax+B}{x^2+2x+2}+\frac{C}{x+1}, \tag{14}$$

where A, B and C are constants we have to find. Multiplying both sides of (14) by $(x^2+2x+2)(x+1)$ we obtain

$$\begin{aligned}3x^2+6x+4 &= (Ax+B)(x+1)+C(x^2+2x+2) \\ \iff 3x^2+6x+4 &= Ax^2+Ax+Bx+B+Cx^2+2Cx+2C \\ \iff 3x^2+6x+4 &= (A+C)x^2+(A+B+2C)+B+2C. \end{aligned} \tag{15}$$

Comparing coefficients of powers of x in (15), we get the simultaneous equations $A+C = 3$, $A+B+2C = 6$ and $B+2C = 4$.

Subtracting the third of these from the second yields $A = 2$. Then substituting $A = 2$ in to the first gives $C = 1$ and finally substitution $C = 1$ into the third gives $B = 2$. Note if you can spot these manipuations, you can always do a formal row reduction.
Hence

$$\begin{aligned}\int_0^1 \frac{3x^2+6x+4}{(x^2+2x+2)(x+1)}\,dx &= \int_0^1 \frac{2x+2}{x^2+2x+2} + \frac{1}{x+1}\,dx \\ &= \left[\ln\left(x^2+2x+2\right) + \ln(x+1)\right]_0^1 \\ &= \ln(5) + \ln(2) - (\ln(2) + \ln(1)) \\ &= \ln(5).\end{aligned}$$

Note that if you can't spot the integral of the $\dfrac{2x+2}{x^2+2x+2}$ term then you can use the substitution $u = x^2 + 2x + 2$.

6. (a) The graph of $f(x) = x^7$ lies below the x-axis between $x = -1$ and $x = 0$ and above the x-axis between $x = 0$ and $x = 1$. Thus the required area is

$$\begin{aligned}-\int_{-1}^0 x^7\,dx + \int_0^1 x^7\,dx &= -\left[\frac{1}{8}x^8\right]_{-1}^0 + \left[\frac{1}{8}x^8\right]_0^1 \\ &= -\left(0 - \frac{1}{8}(-1)^8\right) + \left(\frac{1}{8}(1^8) - 0\right) \\ &= \frac{1}{8} + \frac{1}{8} \\ &= \frac{1}{4}.\end{aligned}$$

(b) The graph of $f(x) = \cos(3x)$ lies below the x-axis between the points $x = -\dfrac{\pi}{3}$ and $x = -\dfrac{\pi}{6}$ and above the x-axis between the points $-\dfrac{\pi}{6}$ and 0. Thus the required area is

$$\begin{aligned}&-\int_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} \cos(3x)\,dx + \int_{-\frac{\pi}{6}}^{0} \cos(3x)\,dx \\ &= -\left[\frac{1}{3}\sin(3x)\right]_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} + \left[\frac{1}{3}\sin(3x)\right]_{-\frac{\pi}{6}}^{0} \\ &= -\left(\frac{1}{3}\sin\left(-\frac{\pi}{2}\right) - \frac{1}{3}\sin(-\pi)\right) + \left(\frac{1}{3}\sin(0) - \frac{1}{3}\sin\left(-\frac{\pi}{2}\right)\right) \\ &= -\left(\frac{1}{3}(-1) - \frac{1}{3}(0)\right) + \left(\frac{1}{3}(0) - \frac{1}{3}(-1)\right) \\ &= \frac{2}{3}.\end{aligned}$$

(c) The graph of $f(x) = e^{2x}$ lies above the x-axis everyhere. Thus the required area is

$$\int_{-1}^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_{-1}^1 = \frac{1}{2}e^2 - \frac{1}{2}e^{-2} = \frac{1}{2}(e^2 - e^{-2}).$$

(d) Since $f(0) = 2$ and the graph of $f(x) = x^3 - 2x^2 - x + 2$ only crosses the x-axis at $x = -1$ between the points $x = -2$ and $x = 0$, it follows that the graph of $f(x) = x^3 - 2x^2 - x + 2$ lies below the x-axis between the points $x = -2$ and $x = -1$ and above the x-axis between the points $x = -1$ and $x = 0$ (since $f(0) = 2 > 0$ it is above the x-axis at $x = 0$).
Hence the required area is

$$
\begin{aligned}
&-\int_{-2}^{-1} x^3 - 2x^2 - x + 2\,dx + \int_{-1}^{0} x^3 - 2x^2 - x + 2\,dx \\
&= -\left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-2}^{-1} + \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{0} \\
&= -\left[\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2 - \left(4 + \frac{16}{3} - 2 - 4\right)\right] + \left[0 - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right)\right] \\
&= \frac{13}{2}.
\end{aligned}
$$

7. In all these questions we will use the formula $V = \pi \int_a^b f(x)^2\,dx$.

(a) The volume is

$$\pi \int_0^1 (x^2)^2\,dx = \pi \int_0^1 x^4\,dx = \pi \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}\pi.$$

(b) The volume is

$$
\begin{aligned}
\pi \int_0^{\pi} \left(\sqrt{\sin(x)}\right)^2 dx &= \pi \int_0^{\pi} \sin(x)\,dx \\
&= \pi\left[-\cos(x)\right]_0^{\pi} \\
&= \pi[-\cos(\pi) - (-\cos(0))] \\
&= \pi(-(-1) - (-1)) \\
&= 2\pi.
\end{aligned}
$$

(c) The volume is

$$
\begin{aligned}
\pi \int_0^1 \left(-e^{-x}\right)^2 dx &= \pi \int_0^1 e^{-2x}\,dx \\
&= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^1 \\
&= \pi\left(-\frac{1}{2}e^{-2} - \left(-\frac{1}{2}e^{0}\right)\right) \\
&= \pi\left(-\frac{1}{2}e^{-2} - \left(-\frac{1}{2}\right)\right) \\
&= \frac{\pi}{2}\left(1 - e^{-2}\right).
\end{aligned}
$$

(d) The volume is

$$\begin{aligned}
\pi \int_0^\pi \sin^2(x)\,dx &= \pi \int_0^\pi \frac{1}{2} + \frac{1}{2}\cos(2x)\,dx \\
&= \pi \left[\frac{1}{2}x + \frac{1}{4}\sin(2x)\right]_0^\pi \\
&= \pi \left(\frac{\pi}{2} + \frac{1}{4}\sin(2\pi) - \left(0 + \frac{1}{4}\sin(0)\right)\right) \\
&= \pi \left(\frac{\pi}{2} + 0 - (0+0)\right) \\
&= \frac{1}{2}\pi^2.
\end{aligned}$$