Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Chapter 5 Solutions

1. (a) Let A be the event 'the card is a black card' and let B be the event 'the card is a face card', so that $P(A \cup B)$ is the probability that the card is either a black card or is a face card. Since there are 26 black cards out of a total of 52 cards, $P(A) = \frac{26}{52}$. Also, there are 12 face cards, so $P(B) = \frac{12}{52}$. Next, since there are 6 black face cards, $P(A \cap B) = \frac{6}{52}$. Hence we have that the probability that the card selected is either a black card or a face card is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{26}{52} + \frac{12}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}.$$

(b) Let A be the event 'the student selected is a first year' and let B be the event 'the student selected is male', so that $P(A \cup B)$ is the probability that the student selected is either a first year or is male. Since there are 57 first years out of a total of 90 students, $P(A) = \frac{57}{90}$. Also, there are $90 - 46 = 44$ males, so $P(B) = \frac{44}{90}$. Next, since there are 23 female first years and there are 57 first years, there are $57 - 23 = 34$ male first years, so that $P(A \cap B) = \frac{34}{90}$. Hence we have that the probability that the student selected is either a first year or is male is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{57}{90} + \frac{44}{90} - \frac{34}{90} = \frac{67}{90}.$$

(c) Let A be the event 'the staff member selected is a nurse' and let B be the event 'the staff member selected is male', so that $P(A \cup B)$ is the probability that the staff member selected is either a nurse or is male. Since there are 11 nurses out of a total of $5 + 11 = 16$ staff members, $P(A) = \frac{11}{16}$. Also, there are 2 male doctors and $11 - 7 = 4$ male nurses, so there are a total of 6 male staff members. Thus $P(B) = \frac{6}{16}$. Next, there are $11 - 7 = 4$ male nurses so that $P(A \cap B) = \frac{4}{16}$. Hence we have that the probability that the staff member selected is either a nurse or is male is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{11}{16} + \frac{6}{16} - \frac{4}{16} = \frac{13}{16}.$$

(d) Let A be the event that the student failed the second test and let B be the event that the student failed the first test. Now, the probability we want

is $P(A|B)$ and we are given in the question that $P(B) = 0.11$ and that $P(A \cap B) = 0.05$.
Hence the probability of a student failing the second test, given that they have failed the first test is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.05}{0.11} = \frac{5}{11}.$$

(e) Let A be the event that the second marble chosen is black and let B be the first marble chosen is white. Now, the probability we want is $P(A|B)$ and we are given in the question that $P(B) = 0.57$ and that $P(A \cap B) = 0.31$.
Hence the probability of selecting a black marble second, given that the first marble drawn was white

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.31}{0.57} = \frac{31}{57}.$$

(f) Let A be the event that a customer likes strawbery ice cream and let B be the event that a customer likes chocolate ice cream. Now, the probability we want is $P(A|B)$ and we are given in the question that $P(B) = 0.65$ and that $P(A \cap B) = 0.4$.
Hence if we pick a customer who likes chocolate ice cream at random, the probability that they also like strawberry ice cream is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.4}{0.65} = \frac{8}{13}.$$

2. (a) Let us call tossing a head a success and let X denote the number of successes we get in thirteen tosses, so that we want to find $P(X = 9)$. Since $n = 13$, $k = 9$, $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$, the required probability is

$$P(X = 9) = \binom{13}{9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^4 = \frac{715}{8192}.$$

(b) Let us call tossing a white ball a success and let X denote the number of successes we get in nine draws, so that we want to find $P(X = 2)$. Since $n = 9$, $k = 2$, $p = \frac{5}{14}$ and $q = 1 - p = \frac{9}{14}$, the required probability is

$$P(X = 2) = \binom{9}{2}\left(\frac{5}{14}\right)^2\left(\frac{9}{14}\right)^7 \simeq 0.208.$$

(c) Let us call tossing a head a success and let X denote the number of successes we get in eleven tosses, so that we want to find $P(X \leqslant 5)$. In this case the probability of a success is $p = 0.5$ and we are tossing a coin eleven times, so $n = 11$. Since we want the probability of getting at most five heads, we look at the $c = 5$ row and the $p = 0.5$ column in the $n = 11$ block. Using the table, we see that the required probability is $P(X \leqslant 5) = 0.5$ (it is an exact probability in this case).

(d) Let us call drawing a black ball a success and let X denote the number of successes we get in ten draws, so that we want to find $P(X \leqslant 6)$. In this case the probability of a success is $p = 0.7$ and we are drawing a ball ten times, so $n = 10$. Since we want the probability of getting at most six black balls, we look at the $c = 6$ row and the $p = 0.7$ column in the $n = 10$ block. Using the table, we see that the required probability is $P(X \leqslant 6) \simeq 0.350$.

(e) Let us call tossing a head a success and let X denote the number of successes we get in fifteen tosses, so that we want to find $P(X \geqslant 10)$. In this case the probability of a success is $p = 0.5$ and we are tossing a coin fifteen times, so $n = 15$. Since the table doesn't directly give us $P(X \geqslant 10)$, we have to use the fact that $P(X \geqslant 10) = 1 - P(X \leqslant 9)$. So we look at the $c = 9$ row and the $p = 0.5$ column in the $n = 15$ block. Hence the required probability is $P(X \geqslant 10) = 1 - P(X \leqslant 9) \simeq 1 - 0.849 = 0.151$.

(f) Let us call drawing a black ball a success and let X denote the number of successes we get in eleven draws, so that we want to find $P(X \geqslant 8)$. In this case the probability of a success is $p = 0.6$ and we are drawing a ball eleven times, so $n = 11$. Since the table doesn't directly give us $P(X \geqslant 8)$, we have to use the fact that $P(X \geqslant 8) = 1 - P(X \leqslant 7)$. So we look at the $c = 7$ row and the $p = 0.6$ column in the $n = 11$ block. Hence the required probability is $P(X \geqslant 8) = 1 - P(X \leqslant 7) \simeq 1 - 0.704 = 0.296$.

3. (a) Here we will assume that this is a Poisson process with parameter $\lambda = \frac{5}{2}$, where in this case we have taken λ to be the average number of bacteria per 125 cm^3 of water. Thus we have to calculate $P(X = 2)$, where X is the number of these bacteria in the test tube containing 125 cm^3 of water. Using $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ with $k = 2$, we have that the required probability is

$$P(X = 2) = \frac{\left(\frac{5}{2}\right)^2 e^{-\frac{5}{2}}}{2!} \simeq 0.257.$$

(b) Here we will assume that this is a Poisson process with parameter $\lambda = \frac{8}{7}$, where in this case we have taken λ to be the average number of births per two days. Thus we have to calculate $P(X = 3)$, where X is the number of births in the town in the next two days. Using $P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$ with $k = 3$, we have that the required probability is

$$P(X = 3) = \frac{\left(\frac{8}{7}\right)^3 e^{-\frac{8}{7}}}{3!} \simeq 0.079.$$

(c) We will assume that this is a Poisson process with $\lambda = 12$, where λ is the average number of customers arriving in the 6 minutes between between 5.30pm and 5.36pm on Fridays. Thus we have to calculate $P(X < 15)$, where X is the number of customers arriving in the given six minutes. However

$P(X < 15) = P(X \leqslant 14)$ and using the Poisson distribution tables we see that the required probability is

$$P(X < 15) = P(X \leqslant 14) \simeq 0.7720.$$

(d) We will assume that this is a Poisson process with $\lambda = 5$, where λ is the average number of cars arriving in the thirty minutes between 5.30pm and 6pm on Sundays. Thus we have to calculate $P(X \leqslant 5)$, where X is the number of cars arriving in the given thirty minutes. Using the Poisson distribution tables we see that the required probability is

$$P(X \leqslant 5) \simeq 0.6160.$$

(e) We will assume that this is a Poisson process with $\lambda = 6$, where λ is the average number of passengers arriving at the bus stop between 3pm and 3.30pm on a Saturday. Thus we have to calculate $P(X \geqslant 5)$, where X is the number of passengers arriving in the given thirty minutes. However $P(X \geqslant 5) = 1 - P(X \leqslant 4)$ and using the Poisson distribution tables we see that the required probability is

$$P(X \geqslant 5) = 1 - P(X \leqslant 4) \simeq 1 - 0.2851 = 0.7149.$$

(f) We will assume that this is a Poisson process with $\lambda = 15$, where λ is the average number of cyclists arriving at UCD between 9am and 9.03am on Mondays. Thus we have to calculate $P(X > 10)$, where X is the number of cyclists arriving in the given three minutes. But $P(X > 10) = 1 - P(X \leqslant 10)$ and using the Poisson distribution tables we see that the required probability is

$$P(X > 10) = 1 - P(X \leqslant 10) \simeq 1 - 0.1185 = 0.8815.$$

4. (a) Here we can go straight to the normal distribution tables to obtain $P(Z \leqslant 2.12) \simeq 0.9830$.

(b) In this case we first have to use $P(Z > a) = P(Z \geqslant a) = 1 - P(Z \leqslant a)$. Then, using the normal distribution tables,

$$P(Z > 0.14) = 1 - P(Z \leqslant 0.14) \simeq 1 - 0.5557 = 0.4443.$$

(c) Here we will use the fact that $P(a \leqslant Z \leqslant b) = P(Z \leqslant b) - P(Z \leqslant a)$. Hence

$$\begin{aligned} P(-2.33 \leqslant Z \leqslant -0.04) &= P(Z \leqslant -0.04) - P(Z \leqslant -2.33) \\ &\simeq 0.4840 - 0.0099 \\ &= 0.4741. \end{aligned}$$

(d) Suppose that X is a normally distributed random variable with mean 55 and standard deviation 12. Then we want to find $P(X < 40) = P(X \leqslant 40)$. Note that in an exam where the marks are only whole numbers then this will not be true, but we are allowed to assume it since we are told in the question that the marks are normally distributed. Now $\mu = 55$ and $\sigma = 12$, so that $\dfrac{40-\mu}{\sigma} = \dfrac{40-55}{12} = -1.25$. Hence $P(X \leqslant 40) = P(Z \leqslant -1.25)$ and using the tables, we find that the probability of a student scoring less than 40 in the exam is approximately 0.1056.

(e) Suppose that X is a normally distributed random variable with mean 170 and standard deviation 8. Then we want to find $P(X > 168) = P(X \geqslant 168)$. Now $\mu = 170$ and $\sigma = 8$, so that $\dfrac{168-\mu}{\sigma} = \dfrac{168-170}{8} = -0.25$. Hence, $P(X \geqslant 168) = P(Z \geqslant -0.25)$. Next we use $P(Z \geqslant a) = 1 - P(Z \leqslant a)$ and the normal distribution tables to see that the probability of a woman chosen at random in Ireland being taller than 168cm is

$$P(Z \geqslant -0.25) = 1 - P(Z \leqslant -0.25) \simeq 1 - 0.4013 = 0.5987.$$

(f) Suppose that X is a normally distributed random variable with mean 1375 and standard deviation 100. Then we want to find

$$P(1200 < X < 1500) = P(1200 \leqslant X \leqslant 1500).$$

Now $\mu = 1375$ and $\sigma = 100$, so that $\dfrac{1200-\mu}{\sigma} = \dfrac{1200-1375}{100} = -1.75$ and $\dfrac{1500-\mu}{\sigma} = \dfrac{1500-1375}{100} = 1.25$. Hence

$$P(1200 < X < 1500) = P(-1.75 \leqslant Z \leqslant 1.25).$$

Next we use $P(a \leqslant Z \leqslant b) = P(Z \leqslant b) - P(z \leqslant a)$ to see that the probability of a randomly selected light bulb of this brand lasting between 1200 hours and 1500 hours is

$$\begin{aligned} P(-1.75 \leqslant Z \leqslant 1.25) &= P(Z \leqslant 1.25) - P(Z \leqslant -1.75) \\ &\simeq 0.8944 - 0.0401 \\ &= 0.8543. \end{aligned}$$