Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Semester 2 2015-2016 Exam Solutions

1. (a)

$$\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & -2 \end{pmatrix}$$

can't be performed since the matrices are not the same size.

$$\begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 3 & -1 \\ -3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ -1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 9 & -3 \\ -9 & -3 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & -10 & 5 \\ 8 & 3 & 3 \end{pmatrix}.$$

(b) $\det\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix} = 3 \times 2 - 1 \times 2 = 4 \neq 0$, so $\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ has an inverse.

$$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{3}{4} \end{pmatrix}.$$

(c)

$$\begin{pmatrix} -1 & -2 \\ 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -7 & 18 \\ 6 & -9 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} -3 & 2 & -1 \end{pmatrix} \text{ can't be performed,}$$

(d)

$$\begin{aligned} \det\begin{pmatrix} 1 & -1 & -3 \\ -2 & -3 & 4 \\ 1 & -2 & -5 \end{pmatrix} =& 1\begin{vmatrix} -3 & 4 \\ -2 & -5 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 4 \\ 1 & -5 \end{vmatrix} + (-3)\begin{vmatrix} -2 & -3 \\ 1 & -2 \end{vmatrix} \\ =& 1(-3 \times (-5) - 4 \times (-2)) + 1(-2 \times (-5) - 4 \times 1) \\ & - 3(-2 \times (-2) - (-3) \times 1) \\ =& 1(23) + 1(6) - 3(7) \\ =& 23 + 6 - 21 \\ =& 8. \end{aligned}$$

(e) We will row reduce the augmented matrix $\begin{pmatrix} 2 & -1 & 1 & 1 \\ 1 & 1 & -1 & 8 \\ -1 & 3 & 2 & -8 \end{pmatrix}$.

$$R1 \leftrightarrow R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 2 & -1 & 1 & 1 \\ -1 & 3 & 2 & -8 \end{pmatrix}$$

$$\begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 + R1 \end{array} \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & -3 & 3 & -15 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$R2 \to -\tfrac{1}{3}R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$R3 \to R3 - 4R2 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 5 & -20 \end{pmatrix}$$

$$R3 \to \tfrac{1}{5}R3 \quad \begin{pmatrix} 1 & 1 & -1 & 8 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

$$\begin{array}{l} R1 \to R1 + R3 \\ R2 \to R2 + R3 \end{array} \quad \begin{pmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

$$R1 \to R1 - R2 \quad \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Hence the solution is $x = 3,\ y = 1,\ z = -4$.

(f) We have

$$\begin{aligned} \det\left[\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} -2-\lambda & 3 \\ -2 & 5-\lambda \end{pmatrix} \\ &= (-2-\lambda)(5-\lambda) - 3(-2) \\ &= \lambda^2 - 3\lambda - 10 + 6 \\ &= \lambda^2 - 3\lambda - 4. \end{aligned}$$

Hence the characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda+1)(\lambda-4) = 0$. Thus the eigenvalues are $\lambda = -1$ and $\lambda = 4$.
We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -1$:

We have $\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -2x+3y \\ -2x+5y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.

Hence we have the equations $-2x + 3y = -x$ and $-2x + 5y = -y$. Both these equations reduce to $x = 3y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

$\lambda = 4$:

We have $\begin{pmatrix} -2 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} -2x+3y \\ -2x+5y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$.

Hence we have the equations $-2x + 3y = 4x$ and $-2x + 5y = 4y$. Both these equations reduce to $y = 2x$, so taking $x = 1$, say, we obtain the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

2. (a)

$$|z| = |2-i| = \sqrt{2^2+(-1)^2} = \sqrt{4+1} = \sqrt{5}, \quad \overline{z} = \overline{2-i} = 2+i,$$
$$\mathrm{Re}(z) = \mathrm{Re}(2-i) = 2, \quad \mathrm{Im}(z) = \mathrm{Im}(2-i) = -1.$$

$$\begin{aligned} z+w &= (2-i)+(4+i) = (2+4)+(-1+1)i = 6 \\ z-w &= (2-i)-(4+i) = (2-4)+(-1-1)i = -2-2i \\ zw &= (2-i)(4+i) = ((2)(4)-(-1)(1)) + ((2)(1)+(-1)(4))i = 9-2i \\ \frac{z}{w} &= \frac{2-i}{4+i} = \frac{2-i}{4+i}\cdot\frac{4-i}{4-i} = \frac{7-6i}{17} = \frac{7}{17} - \frac{6}{17}i \end{aligned}$$

(b) The real part of $-\sqrt{3}+i$ is negative and its imaginary part is positive, so we are in the situation of Figure 6 in the Complex Numbers notes.
Hence the argument of $-\sqrt{3}+i$ is

$$\theta = \pi - \phi = \pi - \tan^{-1}\left(\left|\frac{1}{-\sqrt{3}}\right|\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Also, the magnitude of $-\sqrt{3}+i$ is $r = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$.
Hence, $-\sqrt{3}+i$ in polar form is

$$-\sqrt{3}+i = 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right).$$

To calculate $(-\sqrt{3}+i)^3$ we will use Corollary 2.3.9 from the Complex Numbers notes. That is we will use

$$(r(\cos(n\theta) + i\sin(n\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)),$$

with $r = 2$, $\theta = \dfrac{5\pi}{6}$ and $n = 3$.

Hence

$$\begin{aligned}
\left(-\sqrt{3}+i\right)^3 &= \left(2\left(\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right)\right)^3 \\
&= 2^3\left(\cos\left(\frac{15\pi}{6}\right)+i\sin\left(\frac{15\pi}{6}\right)\right) \\
&= 8\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right) \\
&= 8\,(0+i) \\
&= 8i.
\end{aligned}$$

(c) We will use the fact (see P.12 of the Complex Numbers notes) that the nth roots are given by

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right) \qquad k=0,1,\ldots,n-1.$$

In this case we have $r=\sqrt{2}$ and $\theta=\dfrac{3\pi}{4}$ and we are looking for the third roots, so we take $n=3$.
Thus the roots are

$$z_k = \left(\sqrt{2}\right)^{\frac{1}{3}}\left(\cos\left(\frac{3\pi/4}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{3\pi/4}{3}+\frac{2k\pi}{3}\right)\right) \qquad k=0,1,2.$$

That is

$$\begin{aligned}
z_0 &= 2^{\frac{1}{6}}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) \\
z_1 &= 2^{\frac{1}{6}}\left(\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right) \\
z_2 &= 2^{\frac{1}{6}}\left(\cos\left(\frac{19\pi}{12}\right)+i\sin\left(\frac{19\pi}{24}\right)\right)
\end{aligned}$$

3. (a) (i) We first have to find the critical points of f. To do this we will differentiate f and solve the equation $f'(x)=0$.
However $f'(x)=-6x^2+24x+30$, so we solve the equation $-6x^2+24x+30=0$.
Now

$$\begin{aligned}
-6x^2+24x+30=0 &\Leftrightarrow x^2-4x-5=0 \\
&\Leftrightarrow (x+1)(x-5)=0 \\
&\Leftrightarrow x=-1 \text{ or } x=5.
\end{aligned}$$

Thus the critical points are $x=-1$ and $x=5$.
Next, $f''(x)=-12x+24$ and we evaluate $f''(x)$ at each of the critical points.
$f''(-1)=-12(-1)+24=36>0$, so the critical point at $x=-1$ is a local minimum.
$f''(5)=-12(5)+24=-36<0$, so the critical point at $x=5$ is a local maximum.

(ii) We first have to find the critical points of f, regarding it as having domain $\mathbb{R}$. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = 3x^2 - 18x + 15$, so we solve the equation $3x^2 - 18x + 15 = 0$. Now

$$\begin{aligned} 3x^2 - 18x + 15 = 0 &\Leftrightarrow x^2 - 6x + 5 = 0 \\ &\Leftrightarrow (x-1)(x-5) = 0 \\ &\Leftrightarrow x = 1 \text{ or } x = 5. \end{aligned}$$

Thus the critical points are $x = 1$ and $x = 5$.

We can now find where the global maxima and minima of f occur by evaluating it at the endpoints of the domain and at the critical points that lie in the domain. So we evaluate $f(x)$ at $x = 0, 1, 4$.
$f(0) = 3$, $f(1) = 10$ and $f(4) = -17$.
Hence the global maximum of f is 10 attained at $x = 1$ and the global minimum of f is -17 attained at $x = 4$.

(b) With $f(x) = \dfrac{e^{-2x}\sin(3x)}{\ln(x)}$ we have a product in the numerator, so we have to use the product rule before we use the quotient rule.
First let us differentiate $g(x) = e^{-2x}\sin(3x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(e^{-2x}\right)\sin(3x) + e^{-2x}\frac{d}{dx}\left(\sin(3x)\right) \\ &= -2e^{-2x}\sin(3x) + e^{-2x}\left(3\cos(3x)\right) \\ &= e^{-2x}(3\cos(3x) - 2\sin(3x)). \end{aligned}$$

We can now use the quotient rule with $g(x) = e^{-2x}\sin(3x)$ and $h(x) = \ln(x)$. Then

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{e^{-2x}(3\cos(3x) - 2\sin(3x))\ln(x) - e^{-2x}\sin(3x)\cdot\dfrac{1}{x}}{(\ln(x))^2}. \end{aligned}$$

With $g(x) = \cos(2x^3 - 2x^2 - x + 3)$ we will use the chain rule with $u = 2x^3 - 2x^2 - x + 3$ and $y = \cos(u)$ (where we are letting $y = g(x)$). Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= -\sin(u)\left(6x^2 - 4x - 1\right) \\ &= -\left(6x^2 - 4x - 1\right)\sin\left(2x^3 - 2x^2 - x + 3\right). \end{aligned}$$

4. (a) (i) The graph of $f(x) = \cos(3x)$ lies above the x-axis between the points $x = 0$ and $x = \dfrac{\pi}{6}$ and below the x-axis between the points $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$. Thus the required area is

$$\begin{aligned}
&\int_0^{\frac{\pi}{6}} \cos(3x)\,dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos(3x)\,dx \\
&= \left[\frac{1}{3}\sin(3x)\right]_0^{\frac{\pi}{6}} - \left[\frac{1}{3}\sin(3x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
&= \left(\frac{1}{3}\sin\left(\frac{\pi}{2}\right) - \frac{1}{3}\sin(0)\right) - \left(\frac{1}{3}\sin(\pi) - \frac{1}{3}\sin\left(\frac{\pi}{2}\right)\right) \\
&= \left(\frac{1}{3}(1) - \frac{1}{3}(0)\right) - \left(\frac{1}{3}(0) - \frac{1}{3}(1)\right) \\
&= \frac{2}{3}.
\end{aligned}$$

(ii) Using the formula $V = \pi \displaystyle\int_a^b f(x)^2\,dx$, the volume is

$$\begin{aligned}
V &= \pi \int_{-1}^{1} 1 - x^2\,dx \\
&= \pi \left[x - \frac{1}{3}x^3\right]_{-1}^{1} \\
&= \pi \left[\left(1 - \frac{1}{3}\right) - \left(-1 - \frac{1}{3}(-1)^3\right)\right] \\
&= \pi \left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] \\
&= \frac{4\pi}{3}.
\end{aligned}$$

(b) (i) Here we use integration by parts.
Let $f(x) = 2x$ and $g'(x) = e^{-3x}$,
so that $f'(x) = 2$ and $g(x) = \dfrac{1}{-3}e^{-3x} = -\dfrac{1}{3}e^{-3x}$.
Hence, using the integration by parts formula,

$$\begin{aligned}
\int 2xe^{-3x}\,dx &= 2x \cdot \left(-\frac{1}{3}e^{-3x}\right) - \int 2 \cdot \left(-\frac{1}{3}e^{-3x}\right)\,dx \\
&= -\frac{2}{3}xe^{-3x} + \int \frac{2}{3}e^{-3x}\,dx \\
&= -\frac{2}{3}xe^{-3x} + \frac{2}{3}\left(-\frac{1}{3}e^{-3x}\right) + c \\
&= -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + c \\
&= -\frac{2}{9}e^{-3x}(3x + 1) + c.
\end{aligned}$$

(ii) Here we use partial fractions.
Since $x^2 - 4 = (x-2)(x+2)$, we let

$$\frac{3x+2}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}, \tag{1}$$

where A and B are constants we have to find.
Multiplying both sides of (1) by $x^2 - 4$ we obtain

$$3x + 2 = A(x+2) + B(x-2). \tag{2}$$

If we let $x = -2$ in (2), we obtain $-4 = -4B$, so that $B = 1$.
If we let $x = 2$ in (2), we obtain $8 = 4A$, so that $A = 2$.
Hence

$$\begin{aligned}\int_3^4 \frac{3x+2}{x^2-4}\,dx &= \int_3^4 \frac{2}{x-2} + \frac{1}{x+2}\,dx \\ &= [2\ln(x-2) + \ln(x+2)]_3^4 \\ &= 2\ln(2) + \ln(6) - (2\ln(1) + \ln(5)) \\ &= 2\ln(2) + \ln(6) - \ln(5) \\ &= \ln\left(\frac{24}{5}\right).\end{aligned}$$

Note that if the student can't 'spot' the integrals, then they can use the substitutions $u = x - 2$ and $u = x + 2$ for the two terms, respectively.

5. (a) Let A be the event that a customer likes strawberry ice cream and let B be the event that a customer likes chocolate ice cream. Now, the probability we want is $P(A|B)$ and we are given in the question that $P(B) = 0.75$ and that $P(A \cap B) = 0.45$. Hence if we pick a customer who likes chocolate ice cream at random, the probability that they also like dark chocolate is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.45}{0.75} = \frac{3}{5}.$$

(b) Suppose that X is a normally distributed random variable with mean 170 and standard deviation 8. Then we want to find $P(X < 174)$. Now $\mu = 174$ and $\sigma = 8$, so that $\frac{174-\mu}{\sigma} = \frac{174-170}{8} = 0.5$. Hence, $P(X < 174) = P(Z < 0.5)$. We can now use the normal distribution tables to see that the probability of a woman chosen at random in Ireland being shorter than 174cm is

$$P(Z < 0.5) \simeq 0.6915.$$

(c) Here we will assume that this is a Poisson process with parameter $\lambda = 4$, where in this case we have taken λ to be the average number of bacteria per 200 cm^3 of water. Thus we have to calculate $P(X \geqslant 4)$, where X is the number of these bacteria in the test tube containing 200 cm^3 of water. However $P(X \geqslant 4) = 1 - P(X \leqslant 3)$ and using the Poisson distribution tables we see that $P(X \leqslant 3) \simeq 0.4335$. Thus the required probability is

$$P(X \geqslant 4) = 1 - P(X \leqslant 3) \simeq 1 - 0.4335 = 0.5665.$$