Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Semester 2 2016-2017 Exam Solutions

1. (a)

$$2\begin{pmatrix} 1 & -2 \\ 3 & -1 \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ 6 & -2 \\ -8 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ -3 & -1 \\ -2 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -3 \\ 9 & -1 \\ -6 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 3 \\ -3 & -1 \end{pmatrix}, \text{ that is } \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ -6 & -2 \end{pmatrix}$$

can't be performed since the matrices are not the same size.

(b) $\det\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix} = -1 \times 3 - 1 \times (-2) = -1 \neq 0$, so $\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$ has an inverse.

$$\begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}^{-1} = \frac{1}{-1}\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix}.$$

(c)

$$\begin{pmatrix} -1 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -1 & 4 & 2 \\ 2 & -3 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 2 & 0 \\ -7 & 18 & 8 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -3 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = (8).$$

(d)

$$
\begin{aligned}
(1,2,-1)\times(-1,2,3) =& \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} \\
=& \begin{vmatrix} 2 & -1 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \hat{k} \\
=& (2\times 3 - (-1)\times 2)\hat{i} - (1\times 3 - (-1)\times(-1))\hat{j} \\
& + (1\times 2 - 2\times(-1))\hat{k} \\
=& 8\hat{i} - 2\hat{j} + 4\hat{k} \\
=& (8,-2,4).
\end{aligned}
$$

(e) We will row reduce the augmented matrix $\left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 2 & 1 & -6 & 0 & 1 & 0 \\ -3 & -1 & 9 & 0 & 0 & 1 \end{array}\right)$.

$$
\begin{array}{l}
R2 \to R2 - 2R1 \\
R3 \to R3 + 3R1
\end{array}
\quad
\left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right)
$$

$$
R2 \to -R2 \quad
\left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 2 & -3 & 3 & 0 & 1 \end{array}\right)
$$

$$
R3 \to R3 - 2R2 \quad
\left(\begin{array}{rrr|rrr} 1 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)
$$

$$
\begin{array}{l}
R1 \to R1 + 4R3 \\
R2 \to R2 + 2R3
\end{array}
\quad
\left(\begin{array}{rrr|rrr} 1 & 1 & 0 & -3 & 8 & 4 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)
$$

$$
R1 \to R1 - R2 \quad
\left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -3 & 5 & 2 \\ 0 & 1 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right)
$$

Hence $\begin{pmatrix} 1 & 1 & -4 \\ 2 & 1 & -6 \\ -3 & -1 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -3 & 5 & 2 \\ 0 & 3 & 2 \\ -1 & 2 & 1 \end{pmatrix}$.

(f) We have

$$\det\left[\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$
$$= (1-\lambda)(2-\lambda) - 2(3)$$
$$= \lambda^2 - 3\lambda + 2 - 6$$
$$= \lambda^2 - 3\lambda - 4.$$

Hence the characteristic equation is $\lambda^2 - 3\lambda - 4 = 0$ or $(\lambda+1)(\lambda-4) = 0$. Thus the eigenvalues are $\lambda = -1$ and $\lambda = 4$.

We will now find the eigenvectors corresponding to these eigenvalues.

$\lambda = -1$:
We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} x+2y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$.
Hence we have the equations $x + 2y = -x$ and $3x + 2y = -y$.
Both these equations reduce to $x = -y$, so taking $y = 1$, say, we obtain the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\lambda = 4$:
We have $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} x+2y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$.
Hence we have the equations $x + 2y = 4x$ and $3x + 2y = 4y$.
Both these equations reduce to $3x = 2y$, so taking $y = 3$, say, we obtain the eigenvector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

2. (a)

$$|z| = |1-2i| = \sqrt{1^2 + (-2)^2} = \sqrt{1+4} = \sqrt{5}, \quad \overline{z} = \overline{1-2i} = 1+2i,$$
$$\text{Re}(z) = \text{Re}(1-2i) = 1, \quad \text{Im}(z) = \text{Im}(1-2i) = -2.$$

$$z + w = (1-2i) + (-2+3i) = (1-2) + (-2+3)i = -1 + i,$$
$$z - w = (1-2i) - (-2+3i) = (1+2) + (-2-3)i = 3 - 5i,$$
$$zw = (1-2i)(-2+3i) = ((1)(-2) - (-2)(3)) + ((1)(3) + (-2)(-2))i = 4 + 7i,$$
$$\frac{z}{w} = \frac{1-2i}{-2+3i} = \frac{1-2i}{-2+3i} \cdot \frac{-2-3i}{-2-3i} = \frac{-8+i}{13} = -\frac{8}{13} + \frac{1}{13}i.$$

(b) The real and imaginary parts of $-\sqrt{3}-i$ are both negative, so we are in the situation of Figure 7 in the Complex Numbers notes. Hence the argument of $-\sqrt{3}-i$ is

$$\theta = \phi - \pi = \tan^{-1}\left(\left|\frac{-1}{-\sqrt{3}}\right|\right) - \pi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}.$$

Also, the magnitude of $-\sqrt{3}-i$ is

$$r = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2.$$

Hence, $-\sqrt{3}-i$ in polar form is

$$-\sqrt{3} - i = 2\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) \text{ or } 2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right).$$

To calculate $(-\sqrt{3}-i)^3$ we will use Corollary 2.3.9 from the Complex Numbers notes. That is we will use

$$(r(\cos(n\theta) + i\sin(n\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)),$$

with $r = 2$, $\theta = \dfrac{7\pi}{6}$ and $n = 3$.
Hence

$$\begin{aligned}
\left(-\sqrt{3} - i\right)^3 &= \left(2\left(\cos\left(\frac{7\pi}{6}\right) + i\sin\left(\frac{7\pi}{6}\right)\right)\right)^3 \\
&= 2^3\left(\cos\left(\frac{21\pi}{6}\right) + i\sin\left(\frac{21\pi}{6}\right)\right) \\
&= 8\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) \\
&= 8\,(0 - i) \\
&= -8i.
\end{aligned}$$

(c) We will use the fact (see P.12 of the Complex Numbers notes) that the nth roots are given by

$$z_k = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right) \qquad k = 0, 1, \ldots, n-1.$$

In this case we have $r = 2$ and $\theta = \dfrac{2\pi}{3}$ and we are looking for the third roots, so we take $n = 3$.
Thus the roots are

$$z_k = 2^{\frac{1}{3}}\left(\cos\left(\frac{2\pi/3}{3} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{2\pi/3}{3} + \frac{2k\pi}{3}\right)\right) \qquad k = 0, 1, 2.$$

That is

$$\begin{aligned} z_0 &= 2^{\frac{1}{3}}\left(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right)\right), \\ z_1 &= 2^{\frac{1}{3}}\left(\cos\left(\frac{8\pi}{9}\right) + i\sin\left(\frac{8\pi}{9}\right)\right), \\ z_2 &= 2^{\frac{1}{3}}\left(\cos\left(\frac{14\pi}{9}\right) + i\sin\left(\frac{14\pi}{9}\right)\right). \end{aligned}$$

3. (a) (i) We first have to find the critical points of f. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = -6x^2 + 24x + 30$, so we solve the equation $-6x^2 - 18x + 24 = 0$. Now

$$\begin{aligned} -6x^2 - 18x + 24 = 0 &\Leftrightarrow x^2 + 3x - 4 = 0 \\ &\Leftrightarrow (x+4)(x-1) = 0 \\ &\Leftrightarrow x = -4 \text{ or } x = 1. \end{aligned}$$

Thus the critical points are $x = -4$ and $x = 1$.
Next, $f''(x) = -12x - 18$ and we evaluate $f''(x)$ at each of the critical points.
$f''(-4) = -12(-4) - 18 = 30 > 0$, so the critical point at $x = -4$ is a local minimum.
$f''(1) = -12(1) - 18 = -30 < 0$, so the critical point at $x = 1$ is a local maximum.

(ii) We first have to find the critical points of f, regarding it as having domain $\mathbb{R}$. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = 3x^2 - 6x - 9$, so we solve the equation $3x^2 - 6x - 9 = 0$. Now

$$\begin{aligned} 3x^2 - 6x - 9 = 0 &\Leftrightarrow x^2 - 2x - 3 = 0 \\ &\Leftrightarrow (x+1)(x-3) = 0 \\ &\Leftrightarrow x = -1 \text{ or } x = 3. \end{aligned}$$

Thus the critical points are $x = -1$ and $x = 3$.

We can now find where the global maxima and minima of f occur by evaluating it at the endpoints of the domain and at the critical points that lie in the domain. So we evaluate $f(x)$ at $x = 0, 3, 4$.
$f(0) = 12$, $f(3) = -15$ and $f(4) = -8$.
Hence the global maximum of f is 12 attained at $x = 0$ and the global minimum of f is -15 attained at $x = 3$.

(b) With $f(x) = \dfrac{\ln(2x)\cos(3x)}{e^{4x}}$ we have a product in the numerator, so we have to use the product rule before we use the quotient rule.

First let us differentiate $g(x) = \ln(2x)\cos(3x)$.

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\ln(2x)\right)\cos(3x) + \ln(2x)\frac{d}{dx}\left(\cos(3x)\right) \\ &= \frac{1}{x}\cos(3x) + \ln(2x)\left(-3\sin(3x)\right) \\ &= \frac{1}{x}\cos(3x) - 3\ln(2x)\sin(3x). \end{aligned}$$

We can now use the quotient rule with $g(x) = \ln(2x)\cos(3x)$ and $h(x) = e^{4x}$. Then

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{\left(\frac{1}{x}\cos(3x) - 3\ln(2x)\sin(3x)\right)e^{4x} - \ln(2x)\cos(3x)\cdot 4e^{4x}}{\left(e^{4x}\right)^2}. \end{aligned}$$

With $g(x) = e^{2x^3 - x^2 + 2x - 4}$ we will use the chain rule with $u = 2x^3 - x^2 + 2x - 4$ and $y = e^u$ (where we are letting $y = g(x)$). Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\cdot\frac{du}{dx} \\ &= e^u\left(6x^2 - 2x + 2\right) \\ &= \left(6x^2 - 2x + 2\right)e^{2x^3 - x^2 + 2x - 4}. \end{aligned}$$

4. (a) (i) The graph of $f(x) = \sin(3x)$ lies below the x-axis between the points $x = -\frac{\pi}{6}$ and $x = 0$ and above the x-axis between the points $x = 0$ and $x = \frac{\pi}{3}$. Thus the required area is

$$\begin{aligned} &-\int_{-\frac{\pi}{6}}^{0} \sin(3x)\,dx + \int_{0}^{\frac{\pi}{3}} \sin(3x)\,dx \\ &= -\left[-\frac{1}{3}\cos(3x)\right]_{-\frac{\pi}{6}}^{0} + \left[-\frac{1}{3}\cos(3x)\right]_{0}^{\frac{\pi}{3}} \\ &= -\left(-\frac{1}{3}\cos(0) - \left(-\frac{1}{3}\cos\left(-\frac{\pi}{2}\right)\right)\right) + \left(-\frac{1}{3}\cos(\pi) - \left(-\frac{1}{3}\cos(0)\right)\right) \\ &= -\left(-\frac{1}{3}(1) - \left(-\frac{1}{3}(0)\right)\right) + \left(-\frac{1}{3}(-1) - \left(-\frac{1}{3}(1)\right)\right) \\ &= 1. \end{aligned}$$

(ii) Using the formula $V = \pi \int_a^b f(x)^2 \, dx$, the volume is

$$\begin{aligned} V &= \pi \int_0^{\frac{\pi}{2}} \cos(x) \, dx \\ &= \pi \left[\sin(x)\right]_0^{\frac{\pi}{2}} \\ &= \pi \left[\sin\left(\frac{\pi}{2}\right) - \sin(0)\right] \\ &= \pi \left[1 - 0\right] \\ &= \pi. \end{aligned}$$

(b) (i) Here we use integration by parts.
Let $f(x) = 2x$ and $g'(x) = \sin(3x)$,
so that $f'(x) = 2$ and $g(x) = -\frac{1}{3}\cos(3x)$.
Hence, using the integration by parts formula,

$$\begin{aligned} \int 2x \sin(3x) \, dx &= 2x \cdot \left(-\frac{1}{3}\cos(3x)\right) - \int 2 \cdot \left(-\frac{1}{3}\cos(3x)\right) \, dx \\ &= -\frac{2}{3}x\cos(3x) + \int \frac{2}{3}\cos(3x) \, dx \\ &= -\frac{2}{3}x\cos(3x) + \frac{2}{3}\left(\frac{1}{3}\sin(3x)\right) + c \\ &= -\frac{2}{3}x\cos(3x) + \frac{2}{9}\sin(3x) + c. \end{aligned}$$

(ii) Here we use integration by substitution.
Let $u = x^2 + 16$, so that $\frac{du}{dx} = 2x$. Then $dx = \frac{du}{du/dx} = \frac{du}{2x}$.
Also, when $x = 0$, $u = 16$ and when $x = 3$, $u = 25$.
Hence

$$\begin{aligned} \int_0^3 \frac{x}{\sqrt{x^2+16}} \, dx &= \int_{16}^{25} \frac{x}{\sqrt{u}} \cdot \frac{du}{2x} \\ &= \int_{16}^{25} \frac{1}{2} u^{-\frac{1}{2}} \, du \\ &= \left[\frac{1}{2}\left(2u^{\frac{1}{2}}\right)\right]_{16}^{25} \\ &= \left[u^{\frac{1}{2}}\right]_{16}^{25} \\ &= 25^{\frac{1}{2}} - 16^{\frac{1}{2}} \\ &= 5 - 4 \\ &= 1. \end{aligned}$$

5. (a) Let A be the event 'the student selected is a first year' and let B be the event 'the student selected is male', so that $P(A \cup B)$ is the probability that the student selected is either a first year or is male. Since there are 58 first years out of a total of 95 students, $P(A) = \frac{58}{95}$. Also, there are $95 - 44 = 51$ males, so $P(B) = \frac{51}{95}$. Next, since there are 26 female first years and there are 58 first years, there are $58 - 26 = 32$ male first years, so that $P(A \cap B) = \frac{32}{95}$. Hence we have that the probability that the student selected is either a first year or is male is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{58}{95} + \frac{51}{95} - \frac{32}{95} = \frac{77}{95}.$$

(b) We will assume that this is a Poisson process with $\lambda = 6$, where λ is the average number of cars arriving in the thirty minutes between 5 pm and 5.30 pm on Sundays. Thus we have to calculate $P(X < 7)$, where X is the number of cars arriving in the given thirty minutes. However $P(X < 7) = P(X \leqslant 6)$ and using the Poisson distribution tables we see that the required probability is

$$P(X < 7) = P(X \leqslant 6) \simeq 0.6063.$$

(c) Suppose that X is a normally distributed random variable with mean 1425 and standard deviation 100. Then we want to find

$$P(1300 < X < 1500) = P(1300 \leqslant X \leqslant 1500).$$

Now $\mu = 1425$ and $\sigma = 100$, so that $\frac{1300 - \mu}{\sigma} = \frac{1300 - 1425}{100} = -1.25$ and $\frac{1500 - \mu}{\sigma} = \frac{1500 - 1425}{100} = 0.75$. Hence

$$P(1300 < X < 1500) = P(-1.25 \leqslant Z \leqslant 0.75).$$

Next we use $P(a \leqslant Z \leqslant b) = P(Z \leqslant b) - P(z \leqslant a)$ to see that the probability of a randomly selected light bulb of this brand lasting between 1300 hours and 1500 hours is

$$\begin{aligned} P(-1.25 \leqslant Z \leqslant 0.75) &= P(Z \leqslant 0.75) - P(Z \leqslant -1.25) \\ &\simeq 0.7734 - 0.1056 \\ &= 0.6678. \end{aligned}$$