Access to Science, Engineering and Agriculture: Mathematics 2 MATH00040 Semester 2 2017-2018 Exam Solutions

1. (a)

$$2\begin{pmatrix} 1 & -2 \\ 3 & -1 \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 & -3 \\ -1 & -2 & 1 \end{pmatrix}, \text{ that is } \begin{pmatrix} 2 & -4 \\ 6 & -2 \\ -8 & 0 \end{pmatrix} - \begin{pmatrix} 2 & -1 & -3 \\ -1 & -2 & 1 \end{pmatrix}$$

can't be performed since the matrices are not the same size.

$$\begin{aligned}\begin{pmatrix} 0 & -1 & -1 \\ 1 & 3 & 2 \end{pmatrix} - 2\begin{pmatrix} 1 & 3 & -2 \\ -3 & -1 & 2 \end{pmatrix} &= \begin{pmatrix} 0 & -1 & -1 \\ 1 & 3 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 6 & -4 \\ -6 & -2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -7 & 3 \\ 7 & 5 & -2 \end{pmatrix}.\end{aligned}$$

(b) $\begin{pmatrix} -2 & 3 & 2 \\ -2 & 3 & -2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -3 & 2 \end{pmatrix}$ can't be performed since $\begin{pmatrix} -2 & 3 & 2 \\ -2 & 3 & -2 \end{pmatrix}$ has three columns, while $\begin{pmatrix} -1 & -2 \\ -3 & 2 \end{pmatrix}$ only has two rows.

$$\begin{pmatrix} -1 & 2 & -3 \end{pmatrix}^T \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -3 \\ 2 & -4 & 6 \\ -3 & 6 & -9 \end{pmatrix}$$

(c) The angle between $(1, 2, 1)$ and $(-1, 2, 2)$ is

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{(1,2,1)\cdot(-1,2,2)}{\|(1,2,1)\|\cdot\|(-1,2,2)\|}\right) \\ &= \cos^{-1}\left(\frac{(1)(-1)+(2)(2)+(1)(2)}{\sqrt{1^2+2^2+1^2}\cdot\sqrt{(-1)^2+2^2+2^2}}\right) \\ &= \cos^{-1}\left(\frac{5}{\sqrt{6}\cdot\sqrt{9}}\right) \\ &= \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \\ &\simeq 0.82 \text{ to 2 d.p.}\end{aligned}$$

(d)

$$\det\begin{pmatrix} 1 & -4 & 1 \\ 1 & 6 & -3 \\ -2 & 3 & 0 \end{pmatrix} = 1\begin{vmatrix} 6 & -3 \\ 3 & 0 \end{vmatrix} - (-4)\begin{vmatrix} 1 & -3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 6 \\ -2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= 1(6 \times 0 - (-3) \times 3) + 4(1 \times 0 - (-3) \times (-2)) \\ &\quad + 1(1 \times 3 - 6 \times (-2)) \\ &= 1(9) + 4(-6) + 1(15) \\ &= 9 - 24 + 15 \\ &= 0. \end{aligned}$$

(e) We will row reduce the augmented matrix $\begin{pmatrix} 3 & -11 & -3 & 3 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{pmatrix}$.

$$R1 \to R1 - R2 \qquad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 2 & -6 & -2 & 1 \\ 5 & -17 & -6 & 2 \\ 4 & -8 & 0 & 7 \end{pmatrix}$$

$$\begin{array}{l} R2 \to R2 - 2R1 \\ R3 \to R3 - 5R1 \\ R4 \to R4 - 4R1 \end{array} \qquad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 4 & 0 & -3 \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{pmatrix}$$

$$R2 \to \tfrac{1}{4}R2 \qquad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 8 & -1 & -8 \\ 0 & 12 & 4 & -1 \end{pmatrix}$$

$$\begin{array}{l} R3 \to R3 - 8R2 \\ R4 \to R4 - 12R2 \end{array} \qquad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$$R3 \to -R3 \qquad \begin{pmatrix} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 4 & 8 \end{pmatrix}$$

$$\left(\begin{array}{cccc} 1 & -5 & -1 & 2 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R4 \to R4 - 4R3$$

$$R1 \to R1 + R3 \quad \left(\begin{array}{cccc} 1 & -5 & 0 & 4 \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R1 \to R1 + 5R2 \quad \left(\begin{array}{cccc} 1 & 0 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{3}{4} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence the solution is $x = \dfrac{1}{4},\ y = -\dfrac{3}{4},\ z = 2.$

(f) We have

$$\begin{aligned} \det\left[\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] &= \det\begin{pmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix} \\ &= (3-\lambda)(1-\lambda) - (-1)(1) \\ &= \lambda^2 - 4\lambda + 3 + 1 \\ &= \lambda^2 - 4\lambda + 4. \end{aligned}$$

Hence the characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ or $(\lambda^2 - 2)^2 = 0$ and we have a repeated eigenvalue $\lambda = 2$.

We will now find the eigenvector(s) corresponding to this eigenvalue.

We have $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$, so $\begin{pmatrix} 3x - y \\ x + y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$.

Hence we have the equations $3x - y = 2x$ and $x + y = 2y$.

Both these equations reduce to $x = y$, so taking $y = 1$, say, we obtain the single eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (this matrix has a defect).

2. (a)

$$
\begin{aligned}
|z| = |2-i| = \sqrt{2^2+(-1)^2} = \sqrt{4+1} = \sqrt{5}, \quad \overline{z} = \overline{2-i} = 2+i,\\
\mathrm{Re}(z) = \mathrm{Re}(2-i) = 2, \quad \mathrm{Im}(z) = \mathrm{Im}(2-i) = -1,\\
z+w = (2-i)+(-3+2i) = (2-3)+(-1+2)i = -1+i,\\
z-w = (2-i)-(-3+2i) = (2-(-3))+(-1-2)i = 5-i,\\
zw = (2-i)(-3+2i) = ((2)(-3)-(-1)(2)) + ((2)(2)+(-1)(-3))i = -1+7i,\\
\frac{z}{w} = \frac{2-i}{-3+2i} = \frac{2-i}{-3+2i}\cdot\frac{-3-2i}{-3-2i} = \frac{-8-i}{13} = -\frac{8}{13}-\frac{1}{13}i.
\end{aligned}
$$

(b) The real part of $-2+2i$ is negative and its imaginary part is positive, so we are in the situation of Figure 6 in the Complex Numbers notes.
Hence the argument of $-2+2i$ is

$$\theta = \pi - \phi = \pi - \tan^{-1}\left(\left|\frac{2}{-2}\right|\right) = \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Also, the magnitude, r, of $-2+2i$ is

$$r = \sqrt{(-2)^2+2^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}.$$

Hence, $-2+2i$ in polar form is

$$-2+2i = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right).$$

To calculate $(-2+2i)^4$ we will use Corollary 2.3.9 from the Complex Numbers notes. That is, we will use

$$(r(\cos(n\theta)+i\sin(n\theta)))^n = r^n(\cos(n\theta)+i\sin(n\theta)),$$

with $r = 2\sqrt{2}$, $\theta = \dfrac{3\pi}{4}$ and $n=4$.
Hence

$$
\begin{aligned}
(-2+2i)^4 &= \left(2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)\right)^4\\
&= \left(2\sqrt{2}\right)^4\left(\cos\left(4\cdot\frac{3\pi}{4}\right) + i\sin\left(4\cdot\frac{3\pi}{4}\right)\right)\\
&= 64\left(\cos(3\pi) + i\sin(3\pi)\right)\\
&= 64\left(\cos(\pi) + i\sin(\pi)\right)\\
&= 64\left(-1+0i\right)\\
&= -64.
\end{aligned}
$$

(c) We will use the fact (see P.12 of the Complex Numbers notes) that the nth roots are given by

$$z_k = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) + i \sin\left(\frac{\theta}{n} + \frac{2k\pi}{n} \right) \right) \qquad k = 0, 1, \ldots, n-1.$$

In this case we have $r = 2$ and $\theta = -\dfrac{2\pi}{3}$ and we are looking for the third roots, so we take $n = 3$.
Thus the roots are

$$z_k = 2^{\frac{1}{3}} \left(\cos\left(-\frac{2\pi/3}{3} + \frac{2k\pi}{3} \right) + i \sin\left(-\frac{2\pi/3}{3} + \frac{2k\pi}{3} \right) \right) \qquad k = 0, 1, 2.$$

That is

$$\begin{aligned} z_0 &= 2^{\frac{1}{3}} \left(\cos\left(-\frac{2\pi}{9} \right) + i \sin\left(-\frac{2\pi}{9} \right) \right), \\ z_1 &= 2^{\frac{1}{3}} \left(\cos\left(\frac{4\pi}{9} \right) + i \sin\left(\frac{4\pi}{9} \right) \right), \\ z_2 &= 2^{\frac{1}{3}} \left(\cos\left(\frac{10\pi}{9} \right) + i \sin\left(\frac{10\pi}{9} \right) \right). \end{aligned}$$

3. (a) (i) We first have to find the critical points of f. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = 3x^2 - 6x - 9$, so we solve the equation $3x^2 - 6x - 9 = 0$.
Now

$$\begin{aligned} 3x^2 - 6x - 9 = 0 &\Leftrightarrow x^2 - 2x - 3 = 0 \\ &\Leftrightarrow (x+1)(x-3) = 0 \\ &\Leftrightarrow x = -1 \text{ or } x = 3. \end{aligned}$$

Thus the critical points are $x = -1$ and $x = 3$.
Next, $f''(x) = 6x - 6$ and we evaluate $f''(x)$ at each of the critical points.
$f''(-1) = 6(-1) - 6 = -12 < 0$, so the critical point at $x = -4$ is a local maximum.
$f''(3) = 6(3) - 6 = 12 > 0$, so the critical point at $x = 1$ is a local minimum.

(ii) We first have to find the critical points of f, regarding it as having domain $\mathbb{R}$. To do this we will differentiate f and solve the equation $f'(x) = 0$.
However $f'(x) = -3e^{-3x} + 7$.

Thus we have to solve the equation

$$\begin{aligned}-3e^{-3x}+7=0 &\Leftrightarrow -3e^{-3x}=-7\\ &\Leftrightarrow e^{-3x}=\frac{7}{3}\\ &\Leftrightarrow \ln\left(e^{-3x}\right)=\ln\left(\frac{7}{3}\right)\\ &\Leftrightarrow -3x=\ln\left(\frac{7}{3}\right)\\ &\Leftrightarrow x=-\frac{1}{3}\ln\left(\frac{7}{3}\right).\end{aligned}$$

So there is one critical point of f, that is $x=-\frac{1}{3}\ln\left(\frac{7}{3}\right)$.
We can now find where the global maxima and minima of f occur by evaluating it at the endpoints of the domain and at the critical points that lie in the domain.
So we evaluate $f(x)$ at $x=-1,-\frac{1}{3}\ln\left(\frac{7}{3}\right),0$.
$f(-1)=e^3-7\simeq 13$,
$f\left(-\frac{1}{3}\ln\left(\frac{7}{3}\right)\right)=\exp\left(\ln\left(\frac{7}{3}\right)\right)-\frac{7}{3}\ln\left(\frac{7}{3}\right)=\frac{7}{3}-\frac{7}{3}\ln\left(\frac{7}{3}\right)\simeq 0.4$
and $f(0)=e^0-0=1$.
Hence the global maximum of f is e^3-7 attained at $x=-1$
and the global minimum of f is $\frac{7}{3}-\frac{7}{3}\ln\left(\frac{7}{3}\right)$ attained at $x=-\frac{1}{3}\ln\left(\frac{7}{3}\right)$.

(b) With $f(x)=\dfrac{e^{-2x}\cos(4x)}{\ln(3x)}$ we have a product in the numerator, so we have to use the product rule before we use the quotient rule.
First let us differentiate $g(x)=e^{-2x}\cos(4x)$.

$$\begin{aligned}g'(x)&=\frac{d}{dx}\left(e^{-2x}\right)\cos(4x)+e^{-2x}\frac{d}{dx}\left(\cos(4x)\right)\\ &=-2e^{-2x}\cos(4x)+e^{-2x}\left(-4\sin(4x)\right)\\ &=e^{-2x}(-2\cos(4x)-4\sin(4x)).\end{aligned}$$

We can now use the quotient rule with $g(x)=e^{-2x}\cos(4x)$ and $h(x)=\ln(3x)$.
Then

$$\begin{aligned}f'(x)&=\frac{g'(x)h(x)-g(x)h'(x)}{h(x)^2}\\ &=\frac{e^{-2x}(-2\cos(4x)-4\sin(4x))\ln(3x)-e^{-2x}\cos(4x)\cdot\frac{1}{x}}{\left(\ln(3x)\right)^2}.\end{aligned}$$

With $g(x) = \sin(-2x^3 + 3x^2 + x - 5)$, we will use the chain rule with $u = -2x^3 + 3x^2 + x - 5$ and $y = \sin(u)$ (where we are letting $y = g(x)$). Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \cos(u) \left(-6x^2 + 6x + 1\right) \\ &= \left(-6x^2 + 6x + 1\right) \cos\left(-2x^3 + 3x^2 + x - 5\right). \end{aligned}$$

4. (a) (i) The graph of $f(x) = \cos(2x)$ lies above the x-axis between the points $x = 0$ and $x = \frac{\pi}{4}$, and below the x-axis between the points $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Thus the required area is

$$\begin{aligned} &\int_0^{\frac{\pi}{4}} \cos(2x)\, dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos(2x)\, dx \\ &= \left[\frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\sin(2x)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \left(\frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0)\right) - \left(\frac{1}{2}\sin\left(\frac{3\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right) \\ &= \left(\frac{1}{2}(1) - \frac{1}{2}(0)\right) - \left(\frac{1}{2}(-1) - \frac{1}{2}(1)\right) \\ &= \frac{3}{2}. \end{aligned}$$

(ii) Using the formula $V = \pi \int_a^b f(x)^2\, dx$, the volume is

$$\begin{aligned} V &= \pi \int_0^1 \left(-e^{-x}\right)^2\, dx \\ &= \pi \int_0^1 e^{2x}\, dx \\ &= \pi \left[\frac{1}{2}e^{2x}\right]_0^1 \\ &= \pi \left[\frac{1}{2}e^2 - \frac{1}{2}e^0\right] \\ &= \frac{\pi\left(e^2 - 1\right)}{2}. \end{aligned}$$

(b) (i) Here we will use partial fractions.
Since $x^2 + 9x + 20 = (x+4)(x+5)$, we let

$$\frac{-1}{x^2+9x+20} = \frac{A}{x+4} + \frac{B}{x+5}, \tag{1}$$

where A and B are constants we have to find.
Multiplying both sides of (1) by $x^2 + 9x + 20$ we obtain

$$-1 = A(x+5) + B(x+4). \tag{2}$$

If we let $x = -5$ in (2), we obtain $-1 = -B$, so that $B = 1$.
If we let $x = -4$ in (2), we obtain $-1 = A$, so that $A = -1$.
Hence

$$\begin{aligned}\int \frac{-1}{x^2+9x+20}\,dx &= \int \frac{-1}{x+4} + \frac{1}{x+5}\,dx \\ &= -\ln(x+4) + \ln(x+5) + c.\end{aligned}$$

Note that if you can't 'spot' these integrals, then you can use the substitutions $u = x + 4$ and $u = x + 5$, respectively.

(ii) Here we use integration by substitution.
Let $u = \cos(x)$, so that $\frac{du}{dx} = -\sin(x)$.
Then $dx = \frac{du}{du/dx} = \frac{du}{-\sin(x)}$.
Also, when $x = 0$, $u = 1$ and when $x = \frac{\pi}{2}$, $u = 0$.
Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin(x)e^{\cos(x)}\,dx &= \int_1^0 \sin(x)e^u \cdot \frac{du}{-\sin(x)} \\ &= \int_1^0 -e^u\,du \\ &= [-e^u]_1^0 \\ &= -e^0 - (-e^1) \\ &= -1 + e \\ &= e - 1.\end{aligned}$$

5. (a) Let A be the event that a customer likes strawberry ice cream and let B be the event that a customer likes chocolate ice cream.
Now, the probability we want is $P(A|B)$ and we are given in the question that $P(B) = 0.55$ and that $P(A \cap B) = 0.45$.
Hence if we pick a customer who likes chocolate ice cream at random, the probability that they also like strawberry ice cream is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.45}{0.55} = \frac{9}{11}.$$

(b) Let us call drawing a black ball a success and let X denote the number of successes we get in eleven draws, so that we want to find $P(X \geqslant 8)$.

In this case the probability of a success is $p = 0.6$ and we are drawing a ball eleven times, so $n = 11$.

Since the table doesn't directly give us $P(X \geqslant 8)$, we have to use the fact that $P(X \geqslant 8) = 1 - P(X \leqslant 7)$.

So we look at the $c = 7$ row and the $p = 0.6$ column in the $n = 11$ block.

Hence the required probability is $P(X \geqslant 8) = 1 - P(X \leqslant 7) \simeq 1 - 0.704 = 0.296$.

(c) Suppose that X is a normally distributed random variable with mean 170 and standard deviation 8.

Then we want to find $P(X > 176) = P(X \geqslant 176)$.

Now $\mu = 170$ and $\sigma = 8$, so that $\dfrac{176 - \mu}{\sigma} = \dfrac{176 - 170}{8} = 0.75$.

Hence, $P(X \geqslant 176) = P(Z \geqslant 0.75)$.

Next we use $P(Z \geqslant a) = 1 - P(Z \leqslant a)$ and the normal distribution tables to see that the probability of a woman chosen at random in Ireland being taller than 176cm is

$$P(Z \geqslant 0.75) = 1 - P(Z \leqslant 0.75) \simeq 1 - 0.7734 = 0.2266.$$