“How to Maximize a Function without Really Trying”

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We will prove a famous elementary inequality called **The Rearrangement Inequality**. We will then show that this inequality has some far-reaching consequences!

Motivating Example (Part 1). Banknotes are available in the denominations of EUR5 and EUR10. You are allowed to take 3 banknotes of one type, and 7 banknotes of the other type. How should you choose in order to maximize the amount of money you have?

Answer. Choose 3 EUR5 notes, and 7 EUR10 notes. "Obvious"!

Justification. Because

$$3 \cdot 5 + 7 \cdot 10 > 3 \cdot 10 + 7 \cdot 5 \, .$$

This example motivates the following result.

The Rearrangement Inequality (Case of two variables): Let $a < b$ and $x < y$. Then

$$ax + by > ay + bx \, .$$

Proof: Note that $b - a > 0$ and also $y - x > 0$. Therefore

$$(b - a)(y - x) > 0 \ .$$

Expanding this product yields

$$ax + by - ay - bx > 0 \ ,$$

giving the result.

Motivating Example for the General Case. Banknotes are available in the denominations of EUR5, EUR10 and EUR20. You are allowed to take 3 banknotes of one type, 7 banknotes of a second type, and 9 banknotes of the third type. How should you choose in order to maximize the amount of money you have?

Answer. Choose 3 EUR5 notes, 7 EUR10 notes, and 9 EUR20 notes. Again, "obvious"!

Justification. Because

$$3 \cdot 5 + 7 \cdot 10 + 9 \cdot 20 > 3 \cdot x + 7 \cdot y + 9 \cdot z \ ,$$

where x, y, z is any rearrangement of $5, 10, 20$.

This example motivates the following general result.

The Rearrangement Inequality:

Suppose that

- The n numbers $a_1, a_2, \ldots, a_n$ are in *increasing order*, i.e., $a_1 < a_2 < \cdots < a_n$
- The n numbers $b_1, b_2, \ldots, b_n$ are also in *increasing order*, i.e., $b_1 < b_2 < \cdots < b_n$

If $x_1, x_2, \ldots, x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, \ldots, b_n$, then

(1) $$a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq a_1b_1 + a_2b_2 + \cdots + a_nb_n$$

with equality if and only if the numbers $x_1, x_2, \ldots, x_n$ are in *increasing order*, i.e., if and only if $x_1 = b_1$, $x_1 = b_1$, ..., $x_n = b_n$.

In other words, the maximum of the *mixed sum*

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is equal to the *forward-ordered sum*

$$F = a_1b_1 + a_2b_2 + \cdots + a_nb_n \ .$$

Proof. Suppose we consider any mixed sum

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n \ .$$

Suppose that the arrangement $x_1, x_2, \ldots, x_n$ maximizes the mixed sum. Suppose also that we can find two numbers x_i and x_j such that $a_i < a_j$ but $x_i > x_j$. Suppose we swap x_i with x_j. What happens to the mixed sum?

The mixed sum beforehand equals

$$M = a_1x_1 + a_2x_2 + \cdots + a_ix_i + \cdots + a_jx_j + \cdots + a_nx_n$$

and after the swap equals

$$M' = a_1x_1 + a_2x_2 + \cdots + a_ix_j + \cdots + a_jx_i + \cdots + a_nx_n$$

Does the mixed sum increase? In other words, is $M' > M$? Well, this will be true if

$$a_ix_j + a_jx_i > a_ix_i + a_jx_j \ .$$

But this must be true since

$$(a_j - a_i)(x_i - x_j) > 0 \ .$$

But then the mixed sum after the swap is larger than before the swap. This contradicts our initial assumption that "we can find two numbers x_i and x_j such that $a_i < a_j$ but $x_i > x_j$". If this assumption does not hold, then we must have $x_i < x_j$ whenever $a_i < a_j$.

This shows that the unique arrangement which maximizes the mixed sum is $x_1 = b_1$, $x_2 = b_2$, ..., $x_n = b_n$, i.e., when the numbers $x_1, x_2, \ldots, x_n$ are in *increasing order*. This completes the proof.

Motivating Example for a Related Result. Banknotes are available in the denominations of EUR5, EUR10 and EUR20. You are allowed to take 3 banknotes of one type, 7 banknotes of a second type, and 9 banknotes of the third type. How should you choose in order to **minimize** the amount of money you have?

Answer. Choose 9 EUR5 notes, 7 EUR10 notes, and 3 EUR20 notes.

Corollary to the Rearrangement Inequality:

Suppose that

- The n numbers $a_1, a_2, \ldots, a_n$ are in *increasing order*, i.e., $a_1 < a_2 < \cdots < a_n$
- The n numbers $b_1, b_2, \ldots, b_n$ are also in *increasing order*, i.e., $b_1 < b_2 < \cdots < b_n$

If $x_1, x_2, \ldots, x_n$ is a rearrangement (or permutation) of the numbers $b_1, b_2, \ldots, b_n$, then

(2) $\quad a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$

with equality if and only if $x_1 = b_n$, $x_1 = b_{n-1}$, ..., $x_n = b_1$.

This tells us the minimum of the *mixed sum*

$$M = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is equal to the *reverse-ordered sum*

$$R = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1 \ .$$

Proof of the Corollary to the Rearrangement Inequality:
Applying the Rearrangement Inequality (1) with $-b_n \leq -b_{n-1} \leq -b_1$ in place of $b_1 \leq b_2 \leq \cdots \leq b_n$ we obtain
(3)
$a_1(-x_1)+a_2(-x_2)+\cdots+a_n(-x_n) \leq a_1(-b_n)+a_2(-b_{n-1})+\cdots+a_n(-b_1)$

Here we note that if $x_1, x_2, \ldots, x_n$ is a rearrangement of the numbers $b_1, b_2, \ldots, b_n$, then $-x_1, -x_2, \ldots, -x_n$ is a rearrangement of the numbers $-b_1, -b_2, \ldots, -b_n$.

Simplifying (3) leads to the desired result.

Example: Chebychev's Inequality

Assuming $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, we have

$$R \leq \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n} \leq F\ .$$

Proof: Cyclically rotating the numbers $b_1, b_2, \ldots, b_n$, we get n mixed sums:

$$M_1 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$M_2 = a_1 b_2 + a_2 b_3 + \cdots + a_n b_1$$

$$M_3 = a_1 b_3 + a_2 b_4 + \cdots + a_n b_2$$

$$\vdots$$

$$M_n = a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1}$$

By the rearrangement inequality, each of the n sums lies between R and F. Therefore the average of all of the n sums lies between R and F. But the average of the n sums is

$$\frac{M_1 + M_2 + \cdots + M_n}{n} = \frac{(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n)}{n}\ .$$

This lies between R and F, establishing the desired result.

Exercise. Show that if we substitute $a_1 = b_1 = c_1$, $a_2 = b_2 = c_2$, ..., $a_n = b_n = c_n$ in Chebychev's Inequality we get

$$\sqrt{\frac{c_1^2 + c_2^2 + \cdots + c_n^2}{n}} \geq \frac{c_1 + c_2 + \cdots + c_n}{n},$$

i.e., $\mathrm{RMS} \geq \mathrm{AM}$.

We are now ready to prove the AM-GM inequality. First, let's remind ourselves of this result!

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality:

Suppose we have n positive real numbers $c_1, c_2, \ldots, c_n$. Then

$$\frac{c_1 + c_2 + \cdots + c_n}{n} \geq (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $c_1, c_2, \ldots, c_n$ are equal.

NOTE: The notation $y = x^{\frac{1}{n}}$ means that y is a number whose n-th power is x, i.e., such that $y^n = x$. For example,

- $y = x^{\frac{1}{2}}$ means that $y^2 = x$, i.e., $y = \sqrt{x}$;
- $y = x^{\frac{1}{3}}$ means that $y^3 = x$, i.e., $y = \sqrt[3]{x}$.

Proof of the AM-GM Inequality:

Since all of the numbers $c_1, c_2, \ldots, c_n$ are positive, the geometric mean of these numbers, $\mathrm{GM} = (c_1 c_2 \cdots c_n)^{\frac{1}{n}}$, must also be positive. Let's form

$$a_1 = \frac{c_1}{GM} \; ; \; a_2 = \frac{c_1 c_2}{GM^2} \; ; \; a_3 = \frac{c_1 c_2 c_3}{GM^3} \; ; \cdots \; ; a_n = \frac{c_1 c_2 c_3 \cdots c_n}{GM^n} \; ,$$

and let

$$b_1 = \frac{1}{a_n} \; ; \; b_2 = \frac{1}{a_{n-1}} \; ; \; b_3 = \frac{1}{a_{n-2}} \; ; \cdots \; ; b_n = \frac{1}{a_1} \; .$$

An important observation here is that the ordering of the numbers $b_1, b_2, \ldots, b_n$ is *the same as* that of the numbers $a_1, a_2, \ldots, a_n$. To see this, take the example

$$(a_1, a_2, a_3, a_4, a_5) = (5, 10, 8, 1, 2) \; .$$

In this case

$$(b_1, b_2, b_3, b_4, b_5) = (\frac{1}{2}, \frac{1}{1}, \frac{1}{8}, \frac{1}{10}, \frac{1}{5}) \; .$$

so that

$$a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1 = 5$$

represents the reverse-ordered sum, and is the minimum of any mixed sum.

Applying the Rearrangement Inequality, we find that the mixed sum

$$a_1 b_1 + a_2 b_n + a_3 b_{n-1} + \cdots + a_n b_2$$

is greater than or equal to the reverse-ordered sum

$$a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_nb_1 \ .$$

Working this out we get

$$\frac{c_1}{GM} + \frac{c_2}{GM} + \frac{c_3}{GM} + \cdots \frac{c_n}{GM} \geq n \ ,$$

and simplifying, we get

$$\frac{c_1 + c_2 + \cdots + c_n}{GM} \geq n \ ,$$

or

$$\frac{c_1 + c_2 + \cdots + c_n}{n} \geq GM \ ,$$

in other words, $\mathrm{AM} \geq \mathrm{GM}$.

Exercise. Show that by applying the AM-GM inequality to the numbers $1/c_1, 1/c_2, \ldots, 1/c_n$ we obtain the GM-HM inequality

$$(c_1c_2\cdots c_n)^{\frac{1}{n}} \geq \frac{n}{\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}} \ .$$

Exercise 1: 20 points in the plane are given, none of which are collinear. Divide these into 5 groups. Let N denote the number of triangles with vertices in *different* groups.

How should the points be divided in order to maximize N?

Hint: To get started, let x_1, x_2, x_3, x_4, x_5 denote the number of points in groups $1, 2, 3, 4, 5$, respectively. Then the number of triangles we can form using groups 1, 2 and 3 is $x_1x_2x_3$ (since there are x_1 choices for the vertex from group 1, x_2 choices for the vertex from group 2, and x_3 choices for the vertex from group 3).
Taking into account all of the possible groups for a triangle, we get

$$N = x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + x_3x_4x_5 \ .$$

Next, consider what happens when you take a point out of one group and place it into a different group. Does N increase or decrease?

Exercise 2: Repeat the above problem, but this time you must divide the points into 5 groups **with a different number of points in each group**. How should the points be divided in order to maximize N?

For further reading, click here:

Wikipedia entry on the Rearrangement Inequality