ARITHMETIC PROGRESSIONS
TRAINING PROBLEMS

A strictly increasing sequence $a_1, a_2, \dots, a_n, a_{n+1}, \dots$ is called an **arithmetic progression** or an **arithmetic sequence** if the difference of any two successive members of the sequence is a constant. In other words,

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_{n+1} - a_n = \cdots = r$$

This difference r between any successive terms is called the **(common) difference** or the **ratio** of the arithmetic progression. It is easy to see that the general term a_n has the formula

$$a_n = a_1 + (n-1)r.$$

Also we can easily find the sum S_n of the first n terms of this sequence:

$$S_n = a_1 + a_2 + \cdots + a_n = \frac{n(a_1 + a_n)}{2}.$$

1. Let x, y, z be real numbers such that x^2, y^2, z^2 form an arithmetic progression. Prove that the numbers

$$\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$$

form also an arithmetic progression.

Solution. We have

$$\frac{1}{z+x} - \frac{1}{y+z} = \frac{y-x}{(y+x)(z+x)} = \frac{y^2-x^2}{(y+x)(y+z)(z+x)} = \frac{r}{(y+x)(y+z)(z+x)}$$

and

$$\frac{1}{x+y} - \frac{1}{z+x} = \frac{z-y}{(x+y)(z+x)} = \frac{z^2-y^2}{(y+x)(y+z)(z+x)} = \frac{r}{(y+x)(y+z)(z+x)}.$$

Since x^2, y^2, z^2 is an arithmetic progression, $y^2 - x^2 = z^2 - y^2$ and so

$$\frac{1}{z+x} - \frac{1}{y+z} = \frac{1}{x+y} - \frac{1}{z+x},$$

which means that

$$\frac{1}{y+z}, \frac{1}{z+x}, \frac{1}{x+y}$$

form an arithmetic progression.

2. Let $a_1, a_2, \dots, a_n$ be an arithmetic progression with common difference r. Find in terms of r, and a_1 and a_n the explicit value of sum

$$S = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \cdots + \frac{1}{a_{n-1} a_n}.$$

Solution. Note that

$$\frac{1}{a(a+r)} = \frac{1}{r}\left(\frac{1}{a} - \frac{1}{a+r}\right).$$

So S can be written as a "telescoping sum" in which everything cancels except the first and the last term:

$$S = \frac{1}{r}\left(\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \cdots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right) = \frac{1}{r}\left(\frac{1}{a_1} - \frac{1}{a_n}\right).$$

Remark. In particular, if $a_n = n$ we have:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n} = \frac{n-1}{n}.$$

3. Consider the sequence

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$$

Does there exist an arithmetic progression coposed of this sequences containing 2013 terms?

Solution. Yes! In general, if $n \geq 1$ is a positive integer, the sequence

$$\frac{1}{n!}, \frac{2}{n!}, \frac{3}{n!}, \ldots \frac{n}{n!}$$

is an arithmetic progression.

4. Let A be any set of 19 distinct integers chosen from the arithmetic progression

$$1, 4, 7, \ldots, 100.$$

Prove that there must be two distinct integers in A whose sum is 104. (Putnam competition 1978)

Solution. The pairs of distinct numbers in the arithmetic progression which add to 104 are:

$$(4, 100),\ (7, 97),\ \ldots, (49, 55)$$

There are 16 of these pairs. The numbers 1 and 52 could be in A, but don't appear in any pairs. Apart from these, there are at least $19 - 2 = 17$ numbers in A which all must come from one of these pairs. Since $17 > 16$ and there are 16 pairs, by the Pigeonhole Principle, two of these numbers in A must come from the same pair; so these numbers in A are distinct and add to 104.

5. Prove that if an infinite arithmetic progression of positive integers contains a perfect square, then it contains an infinite number of perfect squares.

Solution. Suppose that n^2 is a perfect square in an infinite arithmetic progression A with common difference $r > 0$. Then

$$(n + r)^2 = n^2 + (2n + r)r$$

is a larger perfect square in A. So for each perfect square in A, there is a larger perfect square in A; applying this statement to the larger perfect square gives us a still larger perfect square in A. Continuing in this manner, we see that there are infinitely many perfect squares in A.

6. Prove that there are no arithmetic progressions of positive integers whose terms are all perfect squares.

Solution. Assume by **contradiction** that there exists positive integers

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \ldots$$

such that

$$a_1^2 < a_2^2 < \cdots < a_n^2 < a_{n+1}^2 < \ldots$$

is an arithmetic progression with common difference r:

$$r = a_2^2 - a_1^2 = a_3^2 - a_2^2 = \cdots = a_n^2 - a_{n-1}^2 = a_{n+1}^2 - a_n^2 = \ldots$$

It follows that

$$(a_n - a_{n-1})(a_n + a_{n-1}) = (a_{n+1} - a_n)(a_{n+1} + a_n), \quad n = 2, 3, 4, \ldots$$

Since $a_{n-1} < a_n < a_{n+1}$ we have $a_{n+1} + a_n > a_n + a_{n-1}$ so the above equality yields

$$a_2 - a_1 > a_3 - a_2 > a_4 - a_3 > \cdots > a_n - a_{n-1} > \cdots > 0$$

which is clearly impossible.

7. (Austrian-Polish Mathematics Competition, 1980)

Three infinite arithmetic progressions are given, whose terms are positive integers. Assuming that each of the numbers $1, 2, 3, 4, 5, 6, 7, 8$ occur in at least one of these progressions, show that 1980 necessarily occurs in one of them.

Solution. Note that if $k|1980$ and if mk and $(m+1)k$ are both in the same infinite arithmetic progression (for some integer m so that $(m+1)k \le 1980$) then 1980 is in that arithmetic progression. Since $1, 2, 3, 4$ all divide 1980, if 1980 is not in any of the three infinite arithmetic progressions, then:
(1). For $1 \le n \le 7$, n and $n+1$ are not in the same arithmetic progression.
(2). If n is $2, 4$ or 6, then n and $n+2$ are not in the same arithmetic progression.
(3). 3 and 6 are not in the same arithmetic progression.
(4). 4 and 8 are not in the same arithmetic progression.
By (2) and (4), the numbers $4, 6$ and 8 are in different arithmetic progressions, so

- let A be the arithmetic progression containing 4,
- let B be the arithmetic progression containing 6, and
- let C be the arithmetic progression containing 8.

By (1), 3 is not in A, and 5 is not in A or B; by (3), 3 is not in B. So 3 and 5 are in C. But C is an infinite arithmetic progression, so 7 is also in C. Since 8 is in C, this contradicts (1). So 1980 must be in at least one of the infinite arithmetic progressions.

8. (International Math Olympiad, 1991) Let $n > 6$ be a positive integer and let
$$1 = a_1 < a_2 < \cdots < a_k$$
be the sequence of all positive integers less than n which are relatively prime with n. Prove that if the sequence $a_1, a_2, \ldots, a_k$ is a non-trivial arithmetic progression, then n is either a prime number or a power of 2.

[A "non-trivial" arithmetic progression has length at least three].

Solution. Let r be the common difference of this arithmetic progression.

Note that

- $r = 1 \iff n$ is coprime with all of $1, 2, \ldots, n-1 \iff n$ is prime; and
- $r = 2 \iff n$ even and coprime with $1, 3, 5, \ldots, n-1 \iff n$ is a power of 2.

Suppose n is neither prime nor a power of 2. Then $r \ge 3$.

Note that $a_1 = 1$ and $a_2 = 1 + r \ge 4$, so 3 is not coprime with n. So $3|n$.

Consider $r \pmod 3$.

Case 1: $r \equiv 0 \pmod 3$. Then $n = 1 + a_k = 2 + (k-1)r \equiv 2 \pmod 3$, so $3 \nmid n$, a contradiction.
Case 2: $r \equiv 1 \pmod 3$. Then $a_3 = 1 + 2r \equiv 0 \pmod 3$, so 3 is a common factor of a_3 and n, a contradiction.
Case 3: $r \equiv 2 \pmod 3$. Then $a_2 = 1 + r \equiv 0 \pmod 3$, so 3 is a common factor of a_2 and n, a contradiction.
In all cases we obtain a contradiction. So n must be either prime or a power of 2.

Homework

1. Let $a_1, a_2, \ldots, a_n$ be an arithmetic progression with common difference r. Find in terms of r and a_1 the explicit value of sums:
   (i). $S = a_1^2 + a_2^2 + \cdots + a_n^2$;
   (ii). $S = \dfrac{1}{a_1a_2a_3} + \dfrac{1}{a_2a_3a_4} + \dfrac{1}{a_3a_4a_5} \cdots + \dfrac{1}{a_{n-2}a_{n-1}a_n}$;
   (iii). $S = \dfrac{a_1 + a_2}{(a_1a_2)^2} + \dfrac{a_2 + a_3}{(a_2a_3)^2} + \dfrac{a_3 + a_4}{(a_3a_4)^2} \cdots + \dfrac{a_{n-1} + a_n}{(a_{n-1}a_n)^2}$.
2. Prove that if an infinite arithmetic progression of positive integers contains a perfect cube, then it contains an infinite number of perfect cubes.
3. Prove that there are no arithmetic progressions of positive integers whose terms are all perfect cubes.