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# Automation, New Technology and Non-Homothetic Preferences\*

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## Abstract

To rationalize a substantial income share of labor despite progressive task automation over the centuries, we present a simple model in which demand moves along a vertically differentiated production structure toward goods of increasing sophistication. Automation of more sophisticated goods requires capital of increasing quality. Quality capital remains scarce along the growth path. This is why labor keeps up a substantial fraction of income. Real capital, however, that is capital measured in units of the quality of some base year, becomes abundant relative to labor. While our model features an entirely different mechanism, we show that its aggregate representation is the one of a neoclassical growth model with labor-augmenting technical change.

**JEL:** E23, E24, E25, J23, J24, O14, O31, O33

**Keywords:** Uzawa's theorem, automation, goods quality, structural change, reallocations, growth, non-homothetic preferences, hierarchical demand

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# 1 Introduction

*Why does labor still earn a substantial share in income despite progressive task automation over the centuries?*<sup>1</sup> *Why does the neoclassical model imply that productivity growth is purely labor-augmenting despite the significant improvements in capital productivity?* To address these questions, we present a simple model that builds on [Acemoglu and Restrepo \(2016, 2017\)](#) (henceforth (A&R)) and introduce a distinction between quality-adjusted and quality-*un*adjusted capital that yields new answers.

In particular, we argue that standard growth models treat capital as a quality-adjusted variable, in the sense that they think of capital in terms of the number of units of capital in some base year. Given this way of treating capital, growth models are correct in asserting that an advanced economy such as the United States uses much more capital in production today than in 1950 (per person). Yet, this view fails to explicitly acknowledge the fact that today’s tasks are hardly comparable to the tasks conducted in an advanced economy in 1950.

Producing one computer today, for example, requires the execution of several advanced tasks that could not be conducted by a machine of 1950. Furthermore, it requires the management of a complex global value chain that puts all the pieces together that are produced by different companies in various locations. The management of such a production process has little in common with the management of production in 1950. One unit of a machine that can produce one computer today is probably much more valuable (in USD) than hundreds (if not thousands) of units of machines that can produce one computer of 1950’s quality. Thus, in per capita terms, we use much more real capital (capital counted in units comparable to the quality of units used in 1950) in production today than in 1950.

Our point is all these extra units of quality-adjusted capital (capital that is comparable to capital in 1950’s units) make it seem like capital is abundant relative to labor. In fact, our model approaches the scarcity of labor from a different angle by leaving capital quality-*un*adjusted. Its basic message is that capital and labor are both scarce: there are relatively few machines that can produce computers in today’s quality. Yet, in 1950’s quality, we use many more machines today than in 1950.

Leaving capital quality-*un*adjusted is appropriate when we think of comparing it to labor (when we think of capital per capita). This is because labor itself is usually thought of in terms of the number of persons or the number of hours worked.<sup>2</sup> Thereby, one unit of today’s labor is treated as equal to one unit of labor in 1950 — a point similar to that of [Young \(2014\)](#). But labor today conducts more advanced tasks that cannot even be handled by today’s machines. Labor, thus, remains quality-*un*adjusted when measured in hours or persons.

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<sup>1</sup>While some recent studies such as [Elsby et al. \(2013\)](#), [Karabarbounis and Neiman \(2014\)](#) and [Piketty and Zucman \(2014\)](#) argue that the U.S. labor share has slightly decreased since the 1980s, other studies argue that this decline is due to measurement problems, see e.g. [Auerbach and Hassett \(2015\)](#) and [Rognlie \(2015\)](#). However, all studies clearly show that the U.S. labor share is still a substantial fraction of GDP despite the ongoing automation of tasks that is described in [Autor \(2015\)](#).

<sup>2</sup>Table 1 shows that the quality adjustments for labor that the BLS makes are quantitatively negligible. According to these data, the quality of work today is still the same as in the late 1980s. Yet, this completely misses the point that more basic tasks such as the one of a waitress requires to interact with customers which makes the job so advanced that even today’s capital cannot easily replace labor here.

More concretely, our model captures these ideas by introducing a quality hierarchy in demand<sup>3</sup> induces continuous reallocations of capital and labor toward more sophisticated goods that become increasingly difficult to produce but provide higher quality.<sup>4,5</sup> Initially, labor has a comparative advantage over capital in the production of these complex goods. Subsequently, however, as technology progresses, capital replaces labor in these tasks and labor moves on to produce even more sophisticated goods. Importantly, automation of more advanced goods requires increasingly sophisticated capital which remains scarce along the growth path.

We derive two parallel aggregate representations of the production side of the economy in our model. The first representation is the neoclassical production function with labor-augmenting technical change. The second representation is the neoclassical production function with factor-neutral technical change. In the standard neoclassical model which is akin to the first representation, capital is treated as a quality-adjusted variable but labor is *not* adjusted for quality.<sup>6</sup> As the quality of both labor and capital improves over time, quality-adjusted capital becomes abundant relative to quality-*un*adjusted labor. The productivity residual that measures how the quality of factors improves therefore indicates that the productivity of labor grows at a higher rate than the productivity of capital. Hence, productivity growth is labor-augmenting. In the second representation capital is quality-*un*adjusted and remains as scarce as quality-*un*adjusted labor. With both factors being treated symmetrically in this representation, productivity is factor neutral.

To explain why labor takes up a substantial fraction in income after hundreds of years of automation, we focus on the same two representations of the neoclassical production function. In the first representation, the productivity growth rate of quality-*un*adjusted labor is higher than the productivity growth rate of quality-adjusted capital, i.e. productivity growth is labor-augmenting. Therefore, the share of labor is stable independent of the elasticity of substitution between capital and labor. The economic interpretation of labor-augmenting technical change, as a result of the asymmetry in quality adjustment in this model, is otherwise difficult. Our second representation offers a more meaningful economic interpretation. Specifically, both quality-*un*adjusted capital and quality-*un*adjusted labor remain equally scarce factors in production along the growth path. This is why they both earn a substantial share in income independent of the elasticity of substitution between capital and labor.

Our paper mainly relates to three different literatures. First, it is a contribution to the litera-

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<sup>3</sup>Buera and Kaboski (2012) and Caron et al. (2014) both link non-homothetic preferences to production characteristics. In particular, they highlight a connection between higher income elasticity goods and the skill-intensity in production. In our model, labor is given by the number of hours (or workers) as in the neoclassical model. Labor is thus *un*adjusted for quality, hence it incorporates quality/skill. Under our framework, production shifts over time to higher-quality goods which are produced by more skilled or more productive labor.

<sup>4</sup>Initially observed by Engel (1857), there is now extensive empirical evidence on non-homothetic preferences, see e.g. Houthakker (1957) and Aguiar and Bils (2015). Our modeling of non-homothetic preferences is based on Laitner (2000), Caselli and Coleman (2001), Matsuyama (2002), Greenwood and Uysal (2005), Foellmi and Zweimüller (2008), Matsuyama (2009) and Fajgelbaum et al. (2011).

<sup>5</sup>A large and growing literature documents the vast improvements in product quality. Contributors to this literature are, among others, Bils and Klenow (2001), Bils (2004), Schott (2004) and Hallak and Schott (2011).

<sup>6</sup>In the neoclassical model, “real” capital (real USD capital) is capital adjusted for quality and variety improvements. By contrast, “real” labor (number of hours/workers) is labor *un*adjusted for quality and variety improvements.

ture that attempts to reconcile the overwhelming empirical evidence on structural change with the empirical evidence on balanced growth.<sup>7</sup> While this literature has featured supply-side mechanisms (e.g. [Ngai and Pissarides \(2007\)](#), [Acemoglu and Guerrieri \(2008\)](#)) and demand side mechanisms (e.g. [Kongsamut et al. \(2001\)](#), [Foellmi and Zweimüller \(2008\)](#)) independently, our paper is in line with recent contributions that attempt to integrate both mechanisms under one framework (e.g. [Buera and Kaboski \(2009\)](#), [Boppart \(2014\)](#), [Herrendorf et al. \(2013\)](#), [Comin et al. \(2015\)](#)). Relative to these contributions our explanation combines a quality hierarchy in demand with industry heterogeneity in labor intensities, factor-neutral productivity growth and automation. In this environment, we derive an endogenously stable labor share in income that is independent of the capital-labor elasticity.

It further differs from those studies in that it seeks to provide a direct microfoundation behind the aggregate production function and is thus closely related to the approaches of [Acemoglu \(2002\)](#) and [Jones \(2005\)](#). Crucially, we differ from these two papers in that we focus on the asymmetry in how the quality of factors is treated by standard frameworks. Our approach stresses the possibility that both capital and labor are scarce factors in the economy. Crucially we extend [Uzawa \(1961\)](#)'s analysis by showing that even with factor-neutral technical change balanced growth can arise independent of the capital-labor elasticity if quality capital remains scarce. Furthermore, we manage to reconcile balanced growth with the substantial improvements in capital productivity that have arguably appeared over time. Our approach resembles that of [Grossman et al. \(2017\)](#) in that we also focus on escaping the straightjacket of Uzawa's theorem. But in contrast to our approach, they focus on endogenous schooling while capital is more complementary with skilled than raw labor.

Second, our paper is a contribution to the recent literature that attempts to reconcile the ongoing automation of tasks with the increase in employment over time. A key variable that this literature indirectly attempts to explain is the substantial share of labor in income despite the continuous replacement of labor-intensive tasks by capital. So far, this literature has offered one central explanation: automation leads to new goods/tasks which sustains employment as outlined by [Olsen and Hemous \(2014\)](#) and A&R, among others. The latter two papers are closest to our approach in that we also assume that labor has a comparative advantage over capital in the production of new tasks/goods. Instead of an endogenous growth mechanism, the driving force in our framework is a combination of non-homothetic preferences and factor-neutral productivity. Besides the simplicity of our model, we differ from A&R, which, as in a standard growth model, exogenously assumes labor-augmenting technical change, in that our mechanism comes from introducing a distinction between quality-adjusted and quality-*un*adjusted capital into an exogenous growth version of their framework.

In what sense is it fundamentally different to say that labor becomes gradually more productive but capital does not (A&R) and that both labor and capital become more productive, but more sophisticated capital is required? While both models can reproduce the same facts, the quality distinction we introduce makes the assumptions behind A&R and other growth models reasonable. Without that distinction, the implication is that the productivity of capital does not improve, while

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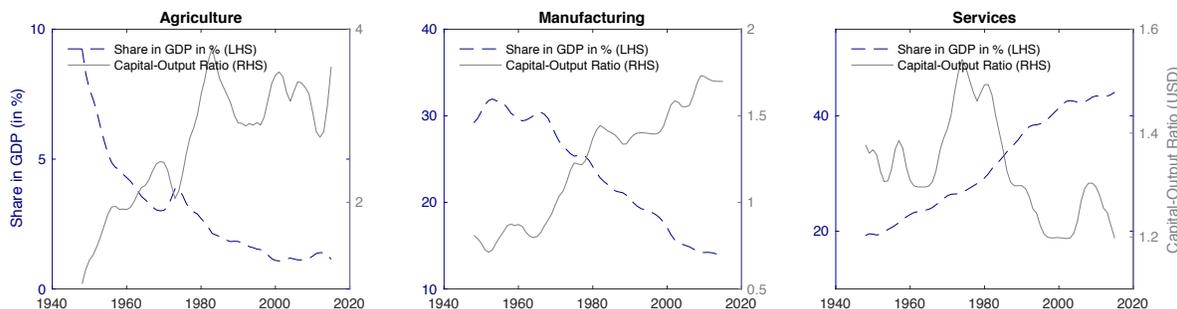
<sup>7</sup>For the evidence on balanced growth see e.g. [Kaldor \(1957\)](#) and [Jones \(2015\)](#).

only the productivity of labor improves. Such an implication is difficult to reconcile, for example, with relatively low productivity growth in labor intensive industries and simply the fact that the productivity of capital has improved over time.

Third, our paper is a contribution to the literature that endeavors to empirically identify the elasticity of substitution and the bias of technical change. As [Diamond et al. \(1978\)](#) and [León-Ledesma et al. \(2010\)](#) point out, the identification is notoriously difficult. Our theoretical contribution has important implications for this literature. Namely, it highlights a crucial asymmetry in standard estimation procedures which leads to biased empirical estimates that are difficult to interpret economically. Allowing for factor-biased technical change, many papers find quality-adjusted capital and quality-*un*adjusted labor to be complements (e.g. [Arrow et al. \(1961\)](#), [Antras \(2004\)](#), [Klump et al. \(2007\)](#), [Oberfeld and Raval \(2014\)](#), [Lawrence \(2015\)](#)). We challenge this interpretation of the data by highlighting the possibility that both quality-*un*adjusted capital and quality-*un*adjusted labor remain equally scarce along the growth path. We theoretically outline that using quality-*un*adjusted capital in the estimation procedure most likely alters the bias of technical change from labor-augmenting to factor-neutral.

## 2 Motivating Empirical Evidence

Figure 1: Non-homothetic preferences, automation and structural change.



Notes: The Figure shows the decline in Agriculture and Manufacturing shares in the U.S. Private Industries' GDP as well as the rise of the Services' share. Notably, in Agriculture, which experiences the fastest GDP share decline, the capital-output ratio increases from under 2 to around 3.7 over time. In Manufacturing, which experiences the second fastest GDP share decline, the capital-output ratio rises from 0.7 to 1.7. In Services, which experiences a rise in its GDP share, the capital-output ratio falls from 1.4 to 1.2. The data are drawn from the U.S. Bureau of Economic Analysis (BEA).

We present two pieces of evidence that motivate the ingredients of our model. First, to motivate non-homothetic preferences and automation, Figure 1 shows the U.S. economy's transition from an agricultural, to a manufacturing, to a services economy. The Figure shows the decline in Agriculture and Manufacturing shares in the U.S. Private Industries' GDP as well as the rise of the Services' share. Notably, in Agriculture, which experiences the fastest GDP share decline, the capital-output ratio increases from under 2 to around 3.7 over time. In Manufacturing, which experiences the second fastest GDP share decline, the capital-output ratio rises from 0.7 to 1.7. In Services, which experiences a rise in its GDP share, the capital-output ratio falls from 1.4 to 1.2. The interpretation

that we incorporate into our model with is that the increased use of capital in the older sectors — Agriculture and Manufacturing — is automation which lowers the relative price of goods produced in these sectors. By contrast, the relatively newer sector Services is and remains labor intensive as it is difficult to automate. When the economy progresses, the demand therefore shifts to the scarce and more difficult to automate Services goods. That is why the share of Services in GDP increases.

Second, to motivate the distinction between quality-adjusted and quality-*un*adjusted capital, Table 1 shows capital and labor productivity estimated from a system of equations consisting of an aggregate CES production function and its first order conditions.

Consider the following CES production function with both labor and capital augmenting technology,

$$Y_t = [\gamma[A_{K,t}K_t]^{\frac{\theta-1}{\theta}} + (1-\gamma)[A_{L,t}l_t]^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}},$$

where  $Y_t$  denotes the amount of real, that is quality-adjusted aggregate output in period  $t$ ;  $A_{K,t}, A_{L,t}$  denote the aggregate capital and labor productivity, respectively;  $\theta$  is the aggregate elasticity of substitution between capital and labor,  $\gamma$  the share of capital in production;  $K_t$  denotes real, that is quality-adjusted, aggregate capital input while  $l_t$  denotes quality-*un*adjusted labor input. The shares of labor and capital in total output are given as

$$\frac{w_t l_t}{P_t Y_t} = (1-\gamma) \left( \frac{A_{L,t} l_t}{Y_t} \right)^{\frac{\theta-1}{\theta}},$$

$$\frac{R_t K_t}{P_t Y_t} = \gamma \left( \frac{A_{K,t} K_t}{Y_t} \right)^{\frac{\theta-1}{\theta}}.$$

We follow León-Ledesma et al. (2010) among others in solving a system from the previous equations to back out the parameters,  $A_{K,t}, A_{L,t}$ . We employ the BLS Private Business Sector data 1987-2016. More specifically, we estimate the parameters from the change over two 10 year periods, i) 1987-1996 and ii) 2007-2016. We rely on 10 year averages to suppress any business cycle inference as our focus is on the longer-term trends. Since we use normalized variables for  $Y_t, K_t, l_t, A_{K,t}, A_{L,t}$  that equal unity in the initial period,  $\gamma = r_0 K_0 / (P_0 Y_0)$  can be backed out from the data directly. Unfortunately, one of the last two equations is redundant as it can be derived as a combination of the other two and, therefore, we effectively are left with two equations and three variables. To overcome this problem, we estimate  $A_{K,t}, A_{L,t}$  under three different assumptions about  $\theta \in \{0.1, 0.5, 3.5\}$ . This range for the elasticity is within the literature bounds of estimates, see e.g. Klump et al. (2007).

Table 1, shows that the labor input per person has barely changed (from 1 to 1.01) over the 20 year period suggesting that quality improvements of labor play no role in the measurement of labor input. By contrast, the amount of capital input, that is real capital per person, or quality-adjusted capital has increased significantly from 1 to 1.6. The labor share has declined by 4 percentage points from 0.66 to 0.62. What is the bias of technical change? Under all three assumptions about  $\theta$ , technical change is labor augmenting, i.e. the technology of labor improves much faster than the technology of capital. Capital productivity even falls if we assume an elasticity of less than unity, i.e.

complementarity between inputs. Again, we argue that this counterintuitive result is caused by the asymmetry in which inputs are thought of and measured. The productivity of capital over this 20 year period has arguably improved (and significantly so). As we cannot remeasure labor and capital we conduct a hypothetical analysis in the following section in which we incorporate the distinction between quality-adjusted and *un*adjusted inputs into a the A&R framework of automation.

Table 1: U.S. Bureau of Labor Statistics (BLS) Private Business Sector, 1987-2016

year	output per capita, $Y$	labor input per capita, $l$	labor share $wl/PY$	capital input per capita, $K$	capital share $RK/PY$
1987-1997	1.00	1.00	0.66	1.00	0.34
2007-2016	1.44	1.01	0.62	1.60	0.38
<b>Varying <math>\theta</math></b>					
...run 1	$\gamma$		$\theta$	$A_K$	$A_L$
1987-1997	0.34		0.50	1.00	1.00
2007-2016	0.34		0.50	0.81	1.52
...run 2	$\gamma$		$\theta$	$A_K$	$A_L$
1987-1997	0.34		3.50	1.00	1.00
2007-2016	0.34		3.50	1.05	1.31
...run 3	$\gamma$		$\theta$	$A_K$	$A_L$
1987-1997	0.34		0.10	1.00	1.00
2007-2016	0.34		0.10	0.89	1.44

Notes:  $A_K$  and  $A_L$  are the authors' estimates.  $\theta$  is assumed. All other data are drawn from the BLS.

### 3 Theoretical Analysis

Our analysis considers a closed economy. In this economy, good  $n \in \mathbb{Z} \equiv \{\dots, -2, -1, 0, 1, 2, \dots\}$  is produced by firm  $n$  in a competitive market. Goods are differentiated by their quality. Higher indexed goods are characterized by a higher quality. Higher quality goods provide greater real output, but are also more difficult to produce. Initially, higher quality goods can only be produced by labor. Subsequently, as technology develops, capital replaces labor in these tasks and labor moves on to produce more sophisticated goods. Importantly, automation of more advanced goods requires more advanced capital. The upper-case variables  $Y_t, Y_{n,t}, P_t, P_{n,t}, R_{n,t}, K_t, K_{n,t}, C_t, I_t, I_{n,t}$  are quality-adjusted, while the lower case variables  $y_t, y_{n,t}, p_t, p_{n,t}, w_{n,t}, r_{n,t}, k_t, k_{n,t}, l_t, l_{n,t}$  are quality-*un*adjusted.

On the demand side, a representative consumer supplies labor inelastically at the aggregate level. Specifically, in our framework, the consumer's preferences are non-homothetic. That is, as the econ-

omy develops, demand shifts to higher quality goods.

### 3.1 Intratemporal Demand

The quality-adjusted output of good  $n$  is given by  $Y_{n,t} = \left(\frac{1}{\Gamma}\right)^n y_{n,t}$ , where  $y_{n,t}$  is the corresponding quality-*unadjusted* output and  $(1/\Gamma)^n$  is the time invariant quality of good  $n$  with  $\Gamma \in (0,1)$ . Aggregate output demand  $Y_t$  combines the individual goods demands  $Y_{n,t}$  according to the constant elasticity of substitution function

$$Y_t = \left[ \sum_{n \in \mathbb{Z}} \gamma_{n,t}^{\frac{1}{\theta}} [Y_{n,t}]^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} \quad (1)$$

where  $\gamma_{n,t}$  is the time-varying weight of good  $n$  in total output with  $\sum_{n \in \mathbb{Z}} \gamma_{n,t} = 1 \quad \forall t$ , and  $\theta > 1$  is the goods elasticity of substitution. The change in the weight of good  $n$  in aggregate output is driven by the level of economic development. In particular, we define

$$\gamma_{n,t} = \frac{e^{-|t-n|}}{z} \quad (2)$$

where  $z = 1 + 2 \sum_{i=1}^{\infty} e^{-i}$  ensures that the weights across goods sum to unity. Intuitively, this definition implies that demand is continuously shifting toward higher quality goods over time. Overall, quality-adjusted output  $Y_t$  can be used for both consumption  $C_t$  and investment  $I_t$ , namely

$$Y_t \equiv C_t + I_t. \quad (3)$$

Solving the intratemporal optimization problem, which entails maximizing equation (1) subject to the usual expenditure constraint

$$E = \sum_{n \in \mathbb{Z}} P_n Y_n, \quad (4)$$

yields the relative demand for quality-adjusted goods

$$\frac{Y_{n,t}}{Y_{-n,t}} = \left[ \frac{\gamma_{n,t}}{\gamma_{-n,t}} \right] \left[ \frac{P_{n,t}}{P_{-n,t}} \right]^{-\theta} \quad (5)$$

where  $P_{-n,t}$  and  $P_{n,t}$  denote the quality-adjusted prices of goods  $-n$  and  $n$  respectively. Given that the weights on goods depend on the level of development, and hence income implicitly, preferences are non-homothetic. The link between quality-adjusted and quality-*unadjusted* prices is straightforward as  $P_n Y_n = p_n y_n$ . Specifically, it is given by

$$P_{n,t} \left( \frac{1}{\Gamma} \right)^n = p_{n,t}. \quad (6)$$

Therefore, the relative demand for quality-*unadjusted* goods is given by

$$\frac{y_{n,t}}{y_{-n,t}} = \left[ \frac{\gamma_{n,t}}{\gamma_{-n,t}} \right] \left[ \frac{p_{n,t}}{p_{-n,t}} \right]^{-\theta} \left[ \frac{\Gamma^n}{\Gamma^{-n}} \right]^{1-\theta}. \quad (7)$$

### 3.2 Firms

The quality-*unadjusted* output of good  $n$ ,  $y_{n,t}$ , is supplied according to the production function

$$y_{n,t} = f_{n,t}(k_{n,t}, l_{n,t}) = \begin{cases} A_t \Gamma^n l_{n,t}, & \text{if } n > t \\ A_t \Gamma^n k_{n,t}, & \text{if } n \leq t \end{cases} \quad (8)$$

where, as before,  $\Gamma \in (0, 1)$  is the parameter governing how quality changes across goods,  $A_t$  is the level of productivity common to all goods at time  $t$ . The evolution of productivity is given exogenously by the process  $A_t = (1 + x)^t$ , where  $x > 0$ . To maintain a tractable analysis, we impose the parameter constraint  $\Gamma^n = 1/(1 + x)^n$ .

Our analysis employs a dichotomous production setup. In particular, higher quality goods production ( $n > t$ ) at any point in time is purely labor-intensive. Conversely, lower quality goods production ( $n \leq t$ ) at any point in time is purely capital-intensive.

The implication of the production structure is that higher quality goods production is eventually automated over time given technological progress, while labor subsequently moves on to produce even higher quality goods. Intuitively, automation of these latter goods is not possible in the early stages of the product life-cycles due to a lag in technological progress. Put differently, the relatively more sophisticated capital required for the production of such goods is not available initially, meaning that labor therefore has to take its place. Finally, equation (8) indicates that a higher quality good, given by a higher  $n$ , is more difficult to produce than a lower quality good, given by a lower  $n$ , at any point in time.

Sectoral quality-adjusted capital accumulation is given by

$$K_{n,t+1} = (1 - \delta)K_{n,t} + I_{n,t}, \quad (9)$$

where  $\delta$  is the capital depreciation rate and  $I_{n,t}$  is sectoral investment. Meanwhile, sectoral quality-*unadjusted* capital accumulation is given by

$$k_{n,t+1} = (1 - \delta)k_{n,t} + \Gamma^n I_{n,t} \quad (10)$$

The interpretation of this equation is the following: the more sophisticated the goods, the more units of real (quality-adjusted) capital are required to constitute one unit of quality-*unadjusted* capital. To simplify the mathematical solution a bit, we take a short cut, using the following equation instead of Eq. (10):

$$k_{n,t+1} = (1 - \delta)k_{n,t} + \Gamma^t I_{n,t}, \quad (11)$$

where  $\Gamma^t = 1/(1 + x)^t$ . The intuition is similar: at a higher stage of development the capital that is used is more sophisticated, thus, more units of real capital are required to constitute one unit of quality-capital. The propositions we derive later can be derived under both setups. The sum of sectoral investments equates to aggregate investment, while the sum of sectoral capital stocks equates to aggregate capital. Specifically, given our production structure, we define aggregate capital as the sum of sectoral capital:

$$I_t = \sum_{n \leq t} I_{n,t} \quad \text{and} \quad K_t = \sum_{n \leq t} K_{n,t} \quad \text{and} \quad k_t = \sum_{n \leq t} k_{n,t}. \quad (12)$$

### 3.3 Intertemporal Demand

The representative consumer maximizes the present discounted value of lifetime utility from consumption

$$U_0 = \sum_{t=0}^{\infty} \beta^t \frac{[C_t]^{1-\phi}}{1-\phi} \quad (13)$$

subject to the standard per period budget constraint

$$P_t C_t + P_t I_t = \sum_{n > t} w_{n,t} l_{n,t} + \sum_{n \leq t} r_{n,t} k_{n,t} \quad (14)$$

where  $\beta$  is the subjective discount factor,  $\phi$  is a parameter governing the intertemporal elasticity of substitution,  $P_t = [\sum_{n \in \mathbb{Z}} \gamma_{n,t} [P_{n,t}]^{1-\theta}]^{1/(1-\theta)}$  is the welfare based quality-adjusted aggregate price index,  $l_{n,t}$  is quality-*un*adjusted sectoral labor,  $k_{n,t}$  is quality-*un*adjusted sectoral capital,  $w_{n,t}$  is the sectoral wage rate, and  $r_{n,t}$  is the sectoral rental rate. The consumer supplies labor inelastically at the aggregate level with  $l_t = 1$ . Labor market clearing implies that the sum of labor supplies allocated across industries equals the aggregate labor supply, namely

$$l_t = \sum_{n > t} l_{n,t}. \quad (15)$$

### 3.4 Equilibrium

In period  $t$ , the representative household maximizes the Lagrangian function

$$\begin{aligned} \mathcal{L} = \sum_{s=0}^{\infty} \beta^{t+s} & \left[ \frac{C_{t+s}^{1-\phi}}{1-\phi} + \lambda_{t+s} \left( \sum_{n > t+s} w_{n,t+s} l_{n,t+s} + \sum_{n \leq t+s} r_{n,t+s} k_{n,t+s} - \sum_{n \leq t+s} P_{t+s} I_{n,t+s} - P_{t+s} C_{t+s} \right) \right. \\ & + \sum_{n \leq t+s} q_{n,t+s} \left( (1-\delta) k_{n,t+s} + \Gamma^{t+s} I_{n,t+s} - k_{n,t+1+s} \right) \\ & \left. + \chi_{t+s} \left( l_{t+s} - \sum_{n > t+s} l_{n,t+s} \right) \right], \end{aligned}$$

where  $\chi_{t+s}$ ,  $q_{n,t+s}$ , and  $\lambda_{t+s}$  denote Lagrange multipliers. Optimization with respect to  $k_{n,t+s+1}$ ,  $I_{n,t+s}$ ,  $C_{t+s}$ , and  $l_{n,t+s}$  respectively yields the first-order conditions

$$\mathcal{L}_{k_{n,t+s+1}} = (1-\delta)\beta q_{n,t+s+1} + \beta \lambda_{t+s+1} r_{n,t+s+1} - q_{n,t+s} = 0, \quad (16)$$

$$\mathcal{L}_{I_{n,t+s}} = -\lambda_{t+s} P_{t+s} + \Gamma^{t+s} q_{n,t+s} = 0, \quad (17)$$

$$\mathcal{L}_{C_{t+s}} = C_{t+s}^{-\phi} - P_{t+s} \lambda_{t+s} = 0, \quad (18)$$

$$\mathcal{L}_{l_{n,t+s}} = \lambda_{t+s} w_{n,t+s} - \chi_{t+s} = 0. \quad (19)$$

The representative firm in sector  $n$  maximizes profits,  $\pi_{n,t}$ , given by

$$\pi_{n,t} = \begin{cases} p_{n,t}y_{n,t} - w_{n,t}l_{n,t}, & \text{if } n > t \\ p_{n,t}y_{n,t} - r_{n,t}k_{n,t}, & \text{if } n \leq t \end{cases} \quad (20)$$

Optimization with respect to  $l_{n,t}$  and  $k_{n,t}$  leads to the first-order conditions

$$w_{n,t} = p_{n,t}\Gamma^n A_t \quad \text{if } n > t \quad (21a)$$

$$r_{n,t} = p_{n,t}\Gamma^n A_t \quad \text{if } n \leq t \quad (21b)$$

As there is a lot happening simultaneously in our model, we highlight our main results in the simplest way possible by comparing two static points in time. One point is characterized by low productivity, state  $t$ , and the other is characterized by high productivity, state  $t+h$ .<sup>8</sup>

Combining (16), (17) and (18) yields the Euler equation which in equilibrium is

$$R_{n,t} = r_{n,t}\Gamma^t = P_t \left( \frac{1}{\beta} - 1 + \delta \right) \quad \forall n \leq t, \quad (22)$$

where  $R_{n,t}$  denotes rental rate of quality-adjusted capital of firm  $n$  in period  $t$  ( $R_{n,t}$  is the real interest rate in the standard neoclassical model). The returns to quality-*un*adjusted capital are equalized across industries, as are the returns quality-*un*adjusted labor. Namely, we have

$$r_{n,t} = r_{-n,t} \quad \forall n, -n \leq t \quad \text{and} \quad w_{n,t} = w_{-n,t} \quad \forall n, -n > t \quad (23)$$

Substituting the first-order conditions from the firm problem, equations (21), into equations (23) leads to an expression for goods prices in equilibrium,

$$\frac{p_{n,t}}{p_{-n,t}} = \frac{\Gamma^{-n}}{\Gamma^n}, \frac{P_{n,t}}{P_{-n,t}} = 1 \quad \forall n, -n > t \quad \text{and} \quad \forall n, -n \leq t \quad (24)$$

Combining the first equation in (24) with equations (7) and (8) yields the sectoral quality-*un*adjusted labor and capital allocations

$$\frac{l_{n,t}}{\sum_{n>t} l_{n,t}} = \frac{l_{n,t}}{l_t} = \frac{\gamma_{n,t}}{\sum_{n>t} \gamma_{n,t}} \quad \forall n, -n > t \quad (25)$$

$$\frac{k_{n,t}}{\sum_{n \leq t} k_{n,t}} = \frac{k_{n,t}}{k_t} = \frac{\gamma_{n,t}}{\sum_{n \leq t} \gamma_{n,t}} \quad \forall n, -n \leq t. \quad (26)$$

To highlight how overall average product quality in the economy evolves with time, we define the industry-weighted average index

$$Q_t = \sum_{n \in \mathbb{Z}} \frac{1}{\Gamma^n} \frac{P_{n,t} Y_{n,t}}{P_t Y_t} \quad (27)$$

which can be rewritten as

$$Q_t = \frac{1}{\Gamma^t} \left[ \frac{1}{1-e^{-1}} \frac{1}{1-\Gamma e^{-1}} \gamma (1/\beta - 1 + \delta)^{1-\theta} + \frac{e^{-1}}{1-e^{-1}} \frac{\Gamma^{-1} e^{-1}}{1-\Gamma^{-1} e^{-1}} (1 - \gamma (1/\beta - 1 + \delta)^{1-\theta}) \right] \quad (28)$$

<sup>8</sup>In Appendix A we show that the main results can be illustrated using a dynamic equilibrium.

where  $\Gamma^{-1}e^{-1} < 1$  is assumed and where  $\gamma = \sum_{n \leq t+s} \gamma_{n,t+s}$  is a constant. Given that the term in brackets is just a positive constant, equation (28) implies that  $Q_{t+h} > Q_t$ .

**Lemma 1 (Quality Evolution).** *Over time, the average quality of goods produced rises. ■*

To highlight the ongoing automation of tasks within our framework. This is fairly simple as it follows directly from the way we have set up the production side of the model. Over time the number of tasks executed by the factor capital expands, i.e.  $B_{t+h} > B_t$  where  $B_t$  denotes the subset of  $\mathbb{Z}$  containing all elements  $n \leq t$ .

**Lemma 2 (Automation).** *Over time, the number of goods produced by capital rises. ■*

Employing equations (1), (8), (25), (26) and quality-adjusted capital in equilibrium,  $K_t = (1/\Gamma^t)k_t$ , we obtain our first representation of aggregate production, namely

$$Y_t = [\gamma^{\frac{1}{\theta}} [K_t]^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}} [A_t l_t]^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}}. \quad (29)$$

We note that in this first presentation there is an asymmetry on the right-hand side of the equation: capital  $K$  is adjusted for quality while labor  $l$  is not.

**Proposition 1 (Aggregate Production - Representation 1).** *Aggregate production takes the form of a Neoclassical Production Function with labor-augmenting productivity. ■*

Turning to our second representation, we simply substitute out quality-adjusted capital for quality-unadjusted capital in equation (29). We thus obtain

$$Y_t = [\gamma^{\frac{1}{\theta}} [A_t k_t]^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}} [A_t l_t]^{\frac{\theta-1}{\theta}}]^{\frac{\theta}{\theta-1}}. \quad (30)$$

We note that in this second representation there is symmetry on the right-hand side of the equation: capital  $k$  is unadjusted for quality and so is labor  $l$ .

**Proposition 2 (Aggregate Production - Representation 2).** *Aggregate production takes the form of a Neoclassical Production Function with factor-neutral productivity. ■*

We now shift focus to quality-unadjusted capital. In particular, we demonstrate that quality-unadjusted capital remains as scarce as quality-unadjusted labor across the two states of nature  $t$  and  $t+h$ . To see this, transform equation (5) to

$$\frac{Y_{K,t}}{Y_t} = \gamma \left[ \frac{P_{K,t}}{P_t} \right]^{-\theta}, \quad (31)$$

where  $Y_{K,t} = \sum_{n \leq t} Y_{n,t} = A_t \sum_{n \leq t} k_{n,t} = A_t k_t = K_t$  and  $P_{K,t} = P_{n,t} \quad \forall n \leq t$ . Inserting the first aggregate representation of production, equation (29), into equation (31) and solving for  $K_t/l_t$

using  $P_{K,t} = P_{n,t} = \Gamma^n p_{n,t} = r_{n,t}/A_t$  and equation (22) yields

$$\frac{K_t}{l_t} = A_t \left( \frac{\left( \gamma^{-\frac{1}{\theta}} [1/\beta - 1 + \delta] \right)^{\theta-1} - \gamma^{\frac{1}{\theta}}}{(1-\gamma)^{\frac{1}{\theta}}} \right)^{\frac{\theta}{1-\theta}}. \quad (32)$$

Therefore, quality-adjusted capital becomes abundant relative to quality-*un*adjusted labor over time, as  $A_{t+h} > A_t$ . As in equilibrium,  $k_t = \Gamma^t K_t$ , and  $A_t = 1/\Gamma^t$ , equation (32) can be rewritten as

$$\frac{k_t}{l_t} = \left( \frac{\left( \gamma^{-\frac{1}{\theta}} [1/\beta - 1 + \delta] \right)^{\theta-1} - \gamma^{\frac{1}{\theta}}}{(1-\gamma)^{\frac{1}{\theta}}} \right)^{\frac{\theta}{1-\theta}} \quad (33)$$

which now corresponds to the aggregate production representation of equation (30). Importantly, in equation (33), quality-*un*adjusted capital does *not* become abundant relative to quality-*un*adjusted labor as we move from state  $t$  to state  $t+h$ , i.e.  $k_t/l_t = k_{t+h}/l_{t+h}$ . Instead, both factors remain scarce.

**Proposition 3 (Scarcity of Quality-*un*adjusted Capital).** *Over time, quality-adjusted capital (real capital) becomes abundant relative to quality-*un*adjusted labor, but quality-*un*adjusted capital remains as scarce as quality-*un*adjusted labor. ■*

Having shown that quality-*un*adjusted capital and quality-*un*adjusted labor remain equally scarce over time, we next concentrate on the labor share in income. Defining the labor income share as 1 minus the capital income share and using equation (31) we obtain

$$\frac{P_{l,t} Y_{l,t}}{P_t Y_t} = 1 - \frac{P_{K,t} Y_{K,t}}{P_t Y_t} = 1 - \gamma \left[ \frac{P_{K,t}}{P_t} \right]^{1-\theta}, \quad (34)$$

where  $P_{l,t} = P_{n,t} \quad \forall n > t$  and  $Y_{l,t} = \sum_{n>t} Y_{n,t} = A_t \sum_{n>t} l_{n,t} = A_t l_t$ . Substituting the Euler equation (22) into the right-hand side of equation (34) and noting that  $P_{K,t} = r_{n,t}/A_t$  gives

$$\frac{P_{l,t} Y_{l,t}}{P_t Y_t} = 1 - \gamma (1/\beta - 1 + \delta)^{1-\theta}. \quad (35)$$

This expression indicates that the share of labor in income depends only on exogenously given parameters. Therefore, the labor income share is stable across the two states  $t$  and  $t+h$ , i.e.  $P_{l,t} Y_{l,t}/(P_t Y_t) = P_{l,t+h} Y_{l,t+h}/(P_{t+h} Y_{t+h})$ .

**Proposition 4 (Share of Labor in Income).** *Over time, the share of quality-*un*adjusted labor in income is stable because quality-*un*adjusted capital remains scarce. ■*

Moving on, we now highlight how labor and capital continuously reallocate across sectors within the economy over time. In order to do so, we define two industry-weighted average quality indexes,  $F_{l,t}$  for the position of labor in the quality ladder and  $F_{k,t}$  for the position of capital in the quality ladder. These two indexes are given by

$$F_{l,t} = \sum_{n>t} n \frac{\gamma_{n,t}}{\sum_{j>t} \gamma_{j,t}} \quad \text{and} \quad F_{k,t} = \sum_{n \leq t} n \frac{\gamma_{n,t}}{\sum_{j \leq t} \gamma_{j,t}}. \quad (36)$$

Applying formulas for geometric series leads to the expressions

$$F_{l,t} = t + \frac{e^{-1}}{1 - e^{-1}} \frac{e^{-1}}{(1 - e^{-1})^2} \quad \text{and} \quad F_{k,t} = t - \frac{1}{1 - e^{-1}} \frac{e^{-1}}{(1 - e^{-1})^2}. \quad (37)$$

Consequently, it is easy to see that labor and capital reallocate along the quality dimension toward higher quality goods as  $F_{l,t+h} > F_{l,t}$  and  $F_{k,t+h} > F_{k,t}$ .

**Proposition 5 (Structural Change).** *Labor and capital continuously reallocate toward higher-quality firms (higher indexed firms) over time. ■*

## 4 Conclusion

Why is the share of labor in income still substantial despite hundreds of years of ongoing automation of tasks? Why does the neoclassical model imply that productivity growth is labor augmenting despite the significant improvements in the productivity of capital? To address these questions, we present a simple model that builds on [Acemoglu and Restrepo \(2016, 2017\)](#) and introduce a distinction between quality-adjusted and quality-*un*adjusted capital that yields new answers.

Our microfounded model features a hierarchical goods demand structure. As the general productivity in the economy improves, demand shifts from lower-quality to higher-quality tasks. Conducting higher quality tasks yields more real output but, at the same time, also requires more resources. Labor and capital continuously reallocate along the quality dimension toward goods of increasing sophistication. Initially, labor has a comparative advantage over capital in the production of higher quality goods. Subsequently, however, as technology progresses, capital replaces labor in these tasks and labor moves on to produce even more sophisticated goods. Notably, automation of more advanced goods requires increasingly sophisticated capital. Such advanced capital remains as scarce as labor along the growth path.

Importantly, our model highlights two parallel aggregate representations of the production side of the economy. The first one is akin to the production function of a neoclassical growth model with labor-augmenting technical change. It shows the asymmetry in how the two factors, capital and labor, are treated by the neoclassical growth model. While capital is quality-adjusted in this representation, labor is not adjusted for quality improvements. Over time, quality-adjusted capital becomes abundant, with quality-*un*adjusted labor remaining relatively scarce. In this case, productivity growth must therefore be labor-augmenting. The second one is akin to a neoclassical model with factor-neutral technical change. It treats both factors symmetrically in the sense that both factors are quality-*un*adjusted. Within this setting, productivity growth is factor neutral as both the quality of capital and labor improve over time. Given that quality capital and quality labor remain equally scarce along the growth path, labor endogenously acquires a substantial share in income.

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## A Appendix - Dynamic Solution

To show that the economy is on a balanced growth path, consider the Euler equation (22) in the dynamic equilibrium:

$$R_t = r_{n,t}\Gamma^t = r_t\Gamma^t = P_t \left( \Gamma \left( \frac{C_t}{C_{t-1}} \right)^\phi \frac{1}{\beta} - 1 + \delta \right), \quad (\text{A.1})$$

where  $r_t$  and  $R_t$  are the interest rates of any industry  $n$  that uses capital. Suppose now that quality-adjusted consumption,  $C$ , grows at a constant rate  $g$ , i.e.  $C_t/C_{t-1} = 1 + g$ . Employing this equation, we can show that the quality-adjusted capital to labor ratio is a constant:

$$\frac{k_t}{l_t} = \left( \frac{\left( \gamma^{-\frac{1}{\theta}} [\Gamma(1+g)^\phi / \beta - 1 + \delta] \right)^{\theta-1} - \gamma^{\frac{1}{\theta}}}{(1-\gamma)^{\frac{1}{\theta}}} \right)^{\frac{\theta}{1-\theta}}. \quad (\text{A.2})$$

Given that we assume that  $l_t$  is constant, it directly follows that  $k_t$  is constant, too. Given that  $k_t$  and  $l_t$  are both constant, the second production function (30) implies that

$$\frac{Y_t}{Y_{t-1}} = \frac{A_t}{A_{t-1}}. \quad (\text{A.3})$$

In other words, quality-adjusted output,  $Y$  grows at the constant rate  $x$ . Given that  $k_t$  is constant, the capital accumulation equation (11) implies that

$$\frac{I_t}{I_{t-1}} = \frac{\Gamma^{t-1}}{\Gamma^t}. \quad (\text{A.4})$$

Thus, quality-adjusted investment,  $I$ , also grows at the constant rate  $x$ . Employing the resource constraint (3), we conclude that quality-adjusted consumption,  $C$ , must also grow at the constant rate  $x$ . Given that we know the growth rate of  $C$ , it is relatively easy to show that the shares of capital and labor, respectively, are constant in the economy in this dynamic equilibrium. Specifically, substituting the Euler equation (A.1) into Eq. (34) yields the constant share

$$\frac{P_{l,t}Y_{l,t}}{P_tY_t} = 1 - \gamma(\Gamma(1+x)^\phi 1/\beta - 1 + \delta)^{1-\theta}. \quad (\text{A.5})$$

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