Mathematical Enrichment

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12 - 1

(Kevin Hutchinson) kevin.hutchinson@ucd.ie

Combinatorics: "Binomial Coefficients"

If $1 \le k \le n$ we define

$$\binom{n}{k} := \frac{n\cdot(n-1)\cdots\cdot(n-k+1)}{1\cdot 2\cdot\cdots\cdot k}$$

"n choose k"

(Also we define $\binom{n}{0} := 1$)

$$\binom{6}{3} = \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 5\cdot 4 = 20$$

$$\binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 10\cdot 3\cdot 7 = 210$$

(If fact these numbers are always integers.)

What do they count?

(I) Take any set/collection of n distinct objects.

eg. $\{1, 2, .., n\}$

$\binom{n}{k}$ = number of subsets of size k.

Eg. $\{a, b, c, d\}$.

2-element subsets: $\{a,b\}, \{a,c\}, \{a,d\}$ $\{b,c\}, \{b,d\}$ $\{c,d\}$

$\binom{4}{2} = \frac{4\cdot 3}{1\cdot 2} = 6$ such subsets.

(2) $\binom{n}{k}$ = # of binary strings of length n with k 1's.

strings of length 4 with 2 1's

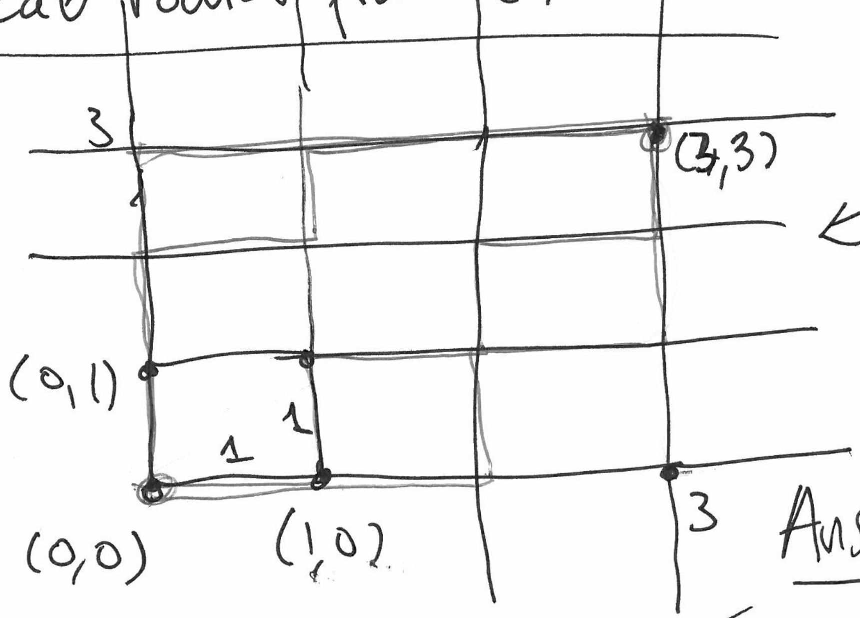
$\binom{4}{2} = 6$ such strings.

strings		subsets
1100		{a, b}
1010		{a, c}
1001	$\longleftrightarrow$	⋮
0110		
0101		
0011		
abcd		

({a, b, d} $\leftrightarrow$ 1101)

(3) "Taxicab routes" from (0,0) to (r, k)

Example



$\binom{6}{3} = 20$

Answer: $\binom{r+k}{k}$

taxicab from (0,0) to (r, k) travel r + k units. with k "ups"

0 = across
1 = up } strings of length r + k with k 1s

Pascal's Triangle

If $1 \leq k \leq n-1$ then $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

[Proof: Algebra exercise.]

$\binom{n}{0} = 1 = \binom{n}{n}$ $\qquad \binom{0}{0} := 1$

$n=0$ 1

$n=1$ 1 1

$n=2$ 1 2 1

$n=3$ 1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$

$(x+y)^3 = \qquad = x^3 + 3x^2y + 3xy^2 + y^3$

$\vdots$

Binomial Theorem

$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-1}xy^{n-1} + y^n$

Exercises (1) $\binom{n}{k} = \binom{n}{n-k}$

(2) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

Binomial Thm. x, y any numbers.

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \cdots + y^n.$$

$x=1,\ y=1$

$$(1+1)^n = \boxed{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}}$$

Claim $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n}$

Pf: Take $x=1,\ y=-1$

$$(1+(-1))^n = 0^n = \binom{n}{0} - \binom{n}{1} + \cdots$$

Exercise

- Show $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$
- Show $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$

[Hint: Think of counting subsets or binary strings]

Claim: If $0 < k < 2^m$ then $\binom{2^m}{k}$ is even.

Proof: $k = 2^s \cdot l$ $\quad$ l odd
$m > s \geq 0$

$$\binom{2^k}{k} = \frac{2^k}{k}\binom{2^k - 1}{k-1} = \frac{2^k}{2^s \cdot l}\binom{2^k - 1}{k-1}$$

$$= \frac{2^{k-s}}{l}\binom{2^k - 1}{k-1} \qquad \boxed{k-s>0}$$

$$\Rightarrow \underset{\text{odd}}{l} \cdot \binom{2^{\cancel{k}m}}{k} = \underbrace{2^{k-s} \cdot \binom{2^k - 1}{k-1}}_{\text{even.}}$$

$$\Rightarrow \binom{2^m}{k} \text{ is even.}$$

Exercise $\quad 0 < k < p^m \quad$ p prime.
Show $p \mid \binom{p^m}{k}$

In how many ways can we distribute 8 lumps of sugar in 5 cups?

Eg

	cup 1	cup 2	cup 3	cup 4	cup 5
dist 1	4	0	4	0	0
dist 2	1	1	1	2	~~5~~3

Solution:

dist 1 $\quad * * * * \mid \; \mid * * * * \mid \; \mid$

dist 2 $\quad * \mid * \mid * \mid * * \mid * * *$

binary strings

dist 1 $\quad 0\,0\,0\,0\,1\,1\,0\,0\,0\,0\,1\,1$

dist 2 $\quad 0\,1\,0\,1\,0\,1\,0\,0\,1\,0\,0\,0$

We can encode ~~dis~~ any distribution as a binary string of length $8+4=12$ with 4 1's

$\therefore$ # of distributions is $\binom{12}{4} = \binom{8+4}{4} = 495.$

How many solutions to

$$X_1 + X_2 + X_3 + X_4 + X_5 = 8$$

where X_i's are integers ≥ 0 ?

Same as previous question!

x_i = # of sugar lumps in cup i.

Answer $\binom{8+4}{4} = 495.$

How many solutions to

$$X_1 + X_2 + \cdots + X_7 = 50 \quad ?$$

$X_i \geq 0$

Answer $\binom{50+6}{6} = \binom{56}{6} = \ldots$

Ex How many solns to

- $x_1 + \cdots + x_k = N$? $\boxed{X_i \geq 0}$

- How many solns to
$$x_1 + \cdots + x_k = N$$
where x_i's are integers ≥ 1 ?

- Same question but x_i's ≥ 4 ?

- A bakery has 8 different kinds of doughnuts.
  How many different 1-dozen boxes can it make?

- How many nondecreasing sequences of length 7 can be made using only numbers from $\{1, 2, .., 20\}$?

Eg. $1, 1, 1, 5, 17, 18, 18$