27 Card Trick & Base 3

Introduction

(Note: It might be useful to do this workshop after the "base systems" workshop)
While there exist many card tricks that rely on various different methods to work, such as sleight of hand and tricks of the senses, few people realise that there are numerous tricks that can be performed using seemingly unrelated areas of mathematics. The 27 card trick is one such trick. Its beauty lies not in the performance. but in the mathematics behind it. It relies on writing numbers in "base 3" and depends on how we arrange the cards into piles.

Aim of the Workshop

The aim of this workshop is to (re–)introduce students to the fundamentals of base systems with emphasis on ternary (base 3) numbers, whilst also providing the opportunity to apply this knowledge to a seemingly unrelated area: Magic Tricks. It is hoped that this workshop will enable students to gain a deeper understanding as to how the decimal system likely developed, whilst also exploring different and entertaining ways of counting.

Learning Outcomes

By the end of this workshop students should be able to:

- Explain, in their own words, what is meant by a base system and why we developed to using the base 10 or decimal system in everyday life
- Recognise the reasons why other base systems might exist
- Convert a decimal number into ternary (i.e. base 3)
- Perform the 27 card trick while understanding the mathematical background.

Materials and Resources

Each pair of students will require: a deck of 27 cards (split a pack of 52 into two suits for each student and give them one joker card each), activity sheets and whiteboard marker.

Key Words

Base system
A way of expressing numbers, using digits or other symbols, in a consistent manner.

Ternary / Base 3
A system of counting which has '3' as its base counting value rather than '10'. It is therefore also commonly referred to as 'base 3'.

27 Card Trick & Base 3: Workshop Outline

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:05)	**Introduction and demonstrate the card trick**	– Demonstrate the trick with one student as a volunteer (see **Appendix – Note 1** for guide) (don't give away the secret of the trick just yet)
10 mins (00:15)	**Counting systems – base 10**	– Introduce counting systems using base 10 (decimal system) [this may be revision, depending on if you have done the Base Systems Workshop] "Why do we think that the decimal system is the one we use in everyday life?"(Main reason being, we have 10 digits) – Explore one example of splitting a number into units, tens, hundreds, thousands etc. and write these as powers of 10 (see **Appendix – Note 1** for example) – Explain the importance of having 10 as the "base" of all the powers – Ask students for their insight to "Why can we write 1 = 100?" – Students can individually attempt **Activity Sheet 1** – While the students are doing Activity Sheet 1 project the blank table from the workshop in the resources onto the whiteboard or draw it out on the board.
10 mins (00:25)	**Mention other counting systems** **Extend to base 3**	– Mention other common bases that can be used such as binary and hexadecimal and give insight into their applications in the world of technology (see bases workshop) – Introduce the concept of base 3 – Explore an example of writing a number in base 3 on the board – Split the students into groups of two or three and give each group one copy of **Activity Sheet 2** from 'A' to 'I' (Alternatively give each group 3 numbers from 0 – 26 to calculate in base 3 and use **Activity Sheet 2**)

SUGGESTED TIME (TOTAL MINS)	ACTIVITY	DESCRIPTION
5 mins (00:30)	**Go through the solutions to activity sheet 2 on the board**	– One student from each group should go to the board and fill in the 3 numbers they calculated in base 3. – Once all groups have gone, ask students to spot any mistakes in the table and fix accordingly. – Ask students if they can see any patterns in the table? – Focus their attention towards the patterns in the columns (the patterns are shown in colour in **Activity Sheet 2 – Card Trick Guide**)
5 mins (00:35)	**Explain how base 3 relates to the trick**	– Link the card trick to the table on the board and explain that to get the important card to a certain position (i.e. the favourite number given by a volunteer), we need to look at how many cards we need above it (see **Appendix – Note 2: teacher's guide**) – **KEY IDEA:** this means we have to subtract 1 from the number given. Relate this back to the demonstration at the start of the lesson. – Explain the idea of labelling • The top pile as the 0th pile, • The middle pile as the 1st pile • And bottom pile as the 2nd pile – Ask "Where have we seen 0s 1s and 2s already?" (On the table on the board)
10 mins (00:45)	**Perform the trick in "annotated mode"**	– Go through the trick again, slowly explaining how you pick up the piles after the volunteer points to the pile with their card in it – **KEY IDEA:** 1st deal — look at the 1s (units) column 2nd deal — look at the 3s column 3rd deal — look at the 9s column – Ask "Where would you put the pile for [different chosen numbers]?"
15 mins (01:00)	**Let the students try the trick themselves**	– In pairs get the students to try the trick on each other – Refer back to the table to split up the number and remember the **TWO KEY IDEAS**: • Always subtract 1 from the number they say at the start • When picking up the piles look at the units first, then the 3s and lastly the 9s – Students might wish to use print outs of the card trick table as necessary (see **Appendix – Note 3**)

27 Card Trick Appendix

Note 1 – Example of base 10 calculations

When we write numbers in the decimal system we are really writing numbers as a sum of units, tens, hundreds, thousands and so on:

E.g.

$$7{,}063 = 7000 + 0 + 60 + 3$$

$$= (7 \times 1000) + (0 \times 100) + (6 \times 10) + (3 \times 1)$$

And by using our knowledge of powers we can write the above as follows

$$= (7 \times 10^3) + (0 \times 10^3) + (6 \times 10^3) + (3 \times 10^3) +$$

So we can see in each bracket above we have our digit (0,1,2,3,4,5,6,7,8,9) multiplied by 10^n where $n \in \mathbb{N}$. So we can write any number as a sum of powers of 10. The fact that we use 10 as our "base" for the power terms is why we call the system "base 10"

Note 2 – How to perform the trick

The goal of the trick is to get the volunteer's chosen card to the position given by their favourite number, without ever knowing what their card was.

- Count out 27 cards from the full deck (or take 2 suits and a joker card [13+13+1 = 27])
- Ask for a volunteer
- Allow the volunteer to choose one of the 27 cards and ask them to memorise it (or show it to the class)
- Ask them to place the card anywhere in the remaining 26 and shuffle the cards
- Begin dealing the 27 cards face up into **3 equal piles** (this will be done 3 times)
- Ask the volunteer to keep an eye out for which pile their card goes into, but not to let you know when they see their card (it must be kept secret until the end!).
- While dealing ask the student "If you had to pick a number between 1 and 27 which would you choose?"
- When they give you the number immediately subtract 1 from it (to yourself) as we need this many cards on top of their chosen card (e.g. if they say '12' – to get the 12th position we need 11 cards on top of that)
- Convert this number to base 3 in your head (this will take practice)

$$11 = (1 \times 9) + (0 \times 3) + (2 \times 1) = (1 \times 32) + (0 \times 31) + (2 \times 30)$$

- Once the cards are dealt, ask the volunteer which pile their card went into. You must now decide where to place this pile for your next deal

– Where you should put this pile of cards is decided by the number of units calculated for each base. This first time you are focusing on and the rule is:

If 0, the pile goes to the **top**

If 1, the pile goes to the **middle**,

If 2, the pile goes to the **bottom**

- E.g. In our earlier example, '11' has **2** units of 3^0 so the pile they point to must go on the **bottom** (the order of the other piles does not matter)

– Deal out the cards again into 3 equal piles and get the student to point to the pile their card went into

– The pile they point to after the second deal must now go to the position (0, 1 or 2) decided by the number of units calculated for each base. So this time you are focusing on 3^1.

- E.g. In our example 11 had **0** units of 3^1 so the pile they point to goes on top of the other **two** piles (the order of the other piles does not matter)

– Deal the card out one last time and ask which pile their card is in and this time place it on top, middle or bottom depending on the number of units calculated for each base. This time you are focusing on the 3^2.

- E.g. In our example there was **1** nine, so the pile they point to goes back in the **middle**

– The trick is now complete so ask them to tell you what card they chose.

– Then ask them to repeat their chosen number between 1 and 27

– Count out that many cards and, if all goes well, the position they chose should be their chosen card!

– As another example, let's imagine that the volunteer says the number 13

- **Step 1:** Subtract 1 from this number, giving us 12
- **Step 2:** Split the number up into base 3
  - 12 contains one 9, one 3 and zero units so we can write it as:
- **Step 3:** When dealing the piles out and asking them to point to the pile their card is in, when putting the piles back together
  - 1st deal look at the **units,** 12 has zero – put their pile back on **top** (0th)
  - 2nd deal look at **threes**, 12 has **one** – put their pile back in the **middle** (1st)
  - 3rd deal look at **nines,** 12 has **one** – put their pile back in the **middle** (2nd)
- Now their chosen card will be in the ***13th position***

– For a video demonstration and further discussion, view the following: https://www.youtube.com/watch?v=l7lP9y7Bb5g

Note 3 – Activity Sheet 2 – Card Trick Table

NUMBER CHOSEN	NUMBER – 1	3^0	3^1	3^2
1	0	0	0	0
2	1	1	0	0
3	2	2	0	0
4	3	0	1	0
5	4	1	1	0
6	5	2	1	0
7	6	0	2	0
8	7	1	2	0
9	8	2	2	0
10	9	0	0	1
11	10	1	0	1
12	11	2	0	1
13	12	0	1	1
14	13	1	1	1
15	14	2	1	1
16	15	0	2	1
17	16	1	2	1
18	17	2	2	1
19	18	0	0	2
20	19	1	0	2
21	20	2	0	2
22	21	0	1	2
23	22	1	1	2
24	23	2	1	2
25	24	0	2	2
26	25	1	2	2
27	26	2	2	2

Remember: 0 = top pile 1 = middle pile 2 = bottom pile

27 Card Trick – Activity 1

$$7{,}603 = 7 \times 10^3 + 6 \times 10^2 + 0 \times 10^1 + 3 \times 10^0$$

Can you write the following numbers in the base 10 format shown in the example above?

1) 78021

2) 26

3) 36.21

Hint for question 3:

$$0.1 = \frac{1}{10} = 10^{-1}$$

$$0.7 = \frac{7}{10} = 7 \times 10^{-1}$$

27 Card Trick – Activity 2 (A)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?
Write them like the first line in the example above (shown in blue)

1) 0 ☐

2) 10 ☐

3) 18 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (B)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?
Write them like the first line in the example above (shown in blue)

1) 4 ☐

2) 16 ☐

3) 21 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (C)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 5 ☐

2) 14 ☐

3) 22 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (D)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 7 ☐

2) 12 ☐

3) 19 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (E)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 6 ☐

2) 13 ☐

3) 20 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (F)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 8 ☐

2) 17 ☐

3) 26 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (G)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 1 ☐

2) 9 ☐

3) 24 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (H)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 3 ☐

2) 15 ☐

3) 25 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2 (I)

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) 2 ☐

2) 11 ☐

3) 23 ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick – Activity 2

Now we move on to working in a different base, Base 3!
In base 3 we will work out the numbers 0–26 by following these rules

$$Number = X \times 3^2 + Y \times 3^1 + Z \times 3^0$$

your X, Y and Z will either have the value 0, 1, or 2

For example:

$$22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$$

$$= 2 \times 9 + 1 \times 3 + 1 \times 1$$

$$= 18 + 3 + 1$$

Remember:

$$3^2 = 9 \quad 3^1 = 3 \quad 3^0 = 1$$

Can you take the following numbers and write them in base 3?

Write them like the first line in the example above (shown in blue)

1) ☐

2) ☐

3) ☐

Once you finish please go to the table on the board and fill in your values.

27 Card Trick: Activity Sheet 2 Solutions Template

NUMBER	3^0	3^1	3^2
0			
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
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