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ON SPORADIC SEQUENCES

BRENDAN ALINQUANT AND ROBERT OSBURN

Abstract. In this note, we prove the last remaining case of the original 15 two-term supercongruence conjectures for sporadic sequences. The proof utilizes a new representation for this sequence (due to Gorodetsky) as the constant term of powers of a Laurent polynomial.

1. Introduction

The original 15 sporadic sequences are integer solutions to specific three-term recurrences. The first six with labels **A**, **B**, **C**, **D**, **E** and **F** were found by Zagier [16], the next six denoted (α), (γ), (δ), (ϵ), (η) and (ζ) were discovered by Almkvist and Zudilin [1] while the final three s_7, s_{10} and s_{18} are due to Cooper [4]. The Apéry numbers for $\zeta(2)$ and $\zeta(3)$ are **D** and (γ), respectively. These sequences are "sporadic" in the sense that they are not terminating, polynomial, hypergeometric or Legendrian solutions [16, Section 3]. Each of the 15 cases has a modular parametrization [5, Tables 1–3], binomial sum representation [14, Tables 1 and 2] and geometric origin [17]. One intriguing aspect of these sequences lies in their arithmetic properties. In [11, 14], the following two-term supercongruences were conjectured.

Conjecture 1.1. *Let $A(n)$ be one of the 15 original sporadic sequences. Then, for all primes $p \geq 5$ and all integers m, $r \geq 1$,*

$$A(mp^r) \equiv A(mp^{r-1}) \pmod{p^{\lambda r}} \tag{1.1}$$

where $\lambda = 3$ except in the cases **B**, **C**, **E**, **F** *and s_{18} in which case $\lambda = 2$.*

Conjecture 1.1 has been established via the techniques in [6] for **A**, **D**, (γ) and s_{10}, [12] for **C**, [13] for **E** and (α), [14] for (ϵ), (η), s_7 and s_{18}, [8] for (ζ), [9] for **B** and [15] for **F**. This leaves the case (δ) which can be expressed as [1]

$$A_\delta(n) = \sum_{k=0}^{n} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{m+k}{n} \frac{(3k)!}{k!^3}. \tag{1.2}$$

The appearance of powers of 3 in (1.2) causes difficulty. For example, after considerable work the $r = 1$ case of Conjecture 1.1 for (δ) was confirmed in [2]. Fortunately, there is an alternative representation for $A_\delta(n)$ as the constant term of $\Lambda(x, y, z)^n$ where

$$\Lambda(x, y, z) = \frac{(x + y - 1)(x + z + 1)(y - x + z)(y - z + 1)}{xyz}.$$

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Let $\boldsymbol{a} = (a_1, \dots, a_\ell)$ be a tuple of nonnegative integers such that $a_1 + \dots + a_\ell = n$ and consider the multinomial coefficient

$$\binom{n}{\boldsymbol{a}} = \binom{n}{a_1, \dots, a_\ell} := \frac{n!}{a_1! \cdots a_\ell!}.$$

In [9, Proposition 3.3], Gorodetsky proved that

$$A_\delta(n) = \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(n)} (-1)^{a_2+b_1+d_3} \binom{n}{\boldsymbol{a}}\binom{n}{\boldsymbol{b}}\binom{n}{\boldsymbol{c}}\binom{n}{\boldsymbol{d}} \tag{1.3}$$

where

$$U(n) = \left\{ (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in \mathbb{Z}_{\geq 0}^{12} \; : \; \begin{matrix} a_1 + a_2 + a_3 = n, & b_1 + c_1 + d_1 = n \\ b_1 + b_2 + b_3 = n, & a_1 + b_2 + d_2 = n \\ c_1 + c_2 + c_3 = n, & a_2 + b_3 + c_2 = n \\ d_1 + d_2 + d_3 = n, & a_3 + c_3 + d_3 = n \end{matrix} \right\}. \tag{1.4}$$

Note that the final condition in (1.4) is the result of taking the sum of the first four equations and then applying the fifth, sixth and seventh equations. We have added this extra equation in order to streamline the subsequent discussion. The purpose of this note is to utilize (1.3) and (1.4) to resolve Conjecture 1.1 in this last remaining case. Our main result is the following.

Theorem 1.2. *For all primes $p \geq 5$ and integers m, $r \geq 1$, we have*

$$A_\delta(mp^r) \equiv A_\delta(mp^{r-1}) \pmod{p^{3r}}.$$

The paper is organized as follows. In Section 2, we present some preliminaries, including two key steps in the proof of Theorem 1.2. The first step (see Proposition 2.3) reduces the proof to considering only those tuples in (1.3) which are not divisible by p while the second step (see Proposition 2.4) decomposes a certain subset of $U(mp^r)$ into disjoint unions of sets. In Section 3, we prove Theorem 1.2. We make three final remarks. First, Straub [15] used (1.3) and (1.4) (without the final condition) to prove (1.1) for the case (δ) with $\lambda = 2$. Second, Cooper [5] has recently investigated sequences that are solutions to four-term recurrences and exhibit several novel features. For example, some of the sequences take values in $\mathbb{Z}[i]$ or $\mathbb{Z}[\sqrt{2}]$ and appear to satisfy "rare" supercongruences [5, Conjectures 11.1–11.5]. Lastly, there is currently no general framework which explains either Conjecture 1.1 or [10, Conjecture 1.3]. For a recent promising development in this direction, see [3].

2. Preliminaries

We first recall the following result [7, Theorem 2.2].

Lemma 2.1. *For primes $p \geq 5$ and integers m, k and r, $s \geq 1$,*

$$\binom{mp^r}{kp^s} \Big/ \binom{mp^{r-1}}{kp^{s-1}} \equiv 1 \pmod{p^{r+s+min(r,s)}}.$$

Note that if $p \nmid k$ and $s \leq r$, then

$$\binom{mp^r}{kp^s} = p^{r-s}\frac{m}{k}\binom{mp^r - 1}{kp^s - 1} \equiv 0 \pmod{p^{r-s}}. \tag{2.1}$$

Let $\sum'$ denote the sum over indices not divisible by p. By considering the parity of these indices and appealing to [14, Lemma 2.2], we obtain the following.

Lemma 2.2. *For primes $p \geq 5$ and integers $s \geq 0$,*

$$\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} \equiv 0 \pmod{p^s}.$$

We now present the two key steps in the proof of Theorem 1.2. For $\beta \in \mathbb{R}$, we write $\beta\boldsymbol{a} = (\beta a_1, \ldots, \beta a_\ell)$. Given $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U(n)$, let

$$B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) := (-1)^{a_2+b_1+d_3} \binom{n}{\boldsymbol{a}}\binom{n}{\boldsymbol{b}}\binom{n}{\boldsymbol{c}}\binom{n}{\boldsymbol{d}}. \tag{2.2}$$

Proposition 2.3. *For all primes $p \geq 5$ and integers m, $r \geq 1$ and $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U(mp^r)$ with $p \mid (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$,*

$$B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \equiv B((\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})/p) \pmod{p^{3r}}. \tag{2.3}$$

Proof. Write $s_1 = \min(\nu_p(a_1), r)$ and $s_2 = \min(\nu_p(a_2), r)$ and suppose $s_1 \geq s_2$. Thus, $a_3 = mp^r - a_1 - a_2$ is divisible by p^{s_2}. It follows from two applications of Lemma 2.1 applied to

$$\binom{mp^r}{\boldsymbol{a}} = \binom{mp^r}{a_1}\binom{mp^r - a_1}{a_2}$$

(note that p^{s_1} divides $mp^r - a_1$) that since $p \mid \boldsymbol{a}$

$$\binom{mp^r}{\boldsymbol{a}} \Big/ \binom{mp^{r-1}}{\boldsymbol{a}/p} \equiv 1 \pmod{p^{s_1+2s_2}}.$$

The same argument applies with $\boldsymbol{a}$ replaced by $\boldsymbol{b}$, $\boldsymbol{c}$ or $\boldsymbol{d}$. Suppose that the value of the quantity $s_1 + 2s_2$ is smallest for $\boldsymbol{a}$ in comparison with those for $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$. Then,

$$\frac{B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})}{B((\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})/p)} \equiv 1 \pmod{p^{s_1+2s_2}}. \tag{2.4}$$

By the constraints defining $U(n)$ in (1.4), we may write

$$B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = (-1)^{a_2+b_1+d_3} \binom{mp^r}{a_1, b_2, d_2}\binom{mp^r}{a_2, b_3, c_2}\binom{mp^r}{a_3, c_3, d_3}\binom{mp^r}{b_1, c_1, d_1}.$$

In particular,

$$\binom{mp^r}{a_1}\binom{mp^r}{a_2}\binom{mp^r}{a_3} \mid B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}). \tag{2.5}$$

By (2.1) and (2.5), we have

$$B(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \equiv 0 \pmod{p^{3r-s_1-2s_2}}. \tag{2.6}$$

Combining (2.4) and (2.6) yields (2.3). □

For a fixed prime $p \geq 5$ and integers m, $r \geq 1$, consider the sets

$$U_{\boldsymbol{ab}}(mp^r) := \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U(mp^r) \,:\, p \nmid \boldsymbol{a} \,\text{and}\, p \nmid \boldsymbol{b}\} \tag{2.7}$$

and

$$U^{(s)}_{\boldsymbol{ab}}(mp^r) := \{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U_{\boldsymbol{ab}}(mp^r) \,:\, s = \min(\nu_p(\boldsymbol{c}), \nu_p(\boldsymbol{d}), r)\}.$$

Clearly, $1 \leq s \leq r$. Furthermore, let $\boldsymbol{x} := (x, -x, 0, 0, -x, x, 0, 0, 0, 0, 0, 0) \in \mathbb{Z}^{12}$,

$$S_{p^s} := \{0 \leq x \leq p^s \,:\, p \nmid x\}$$

and

$$L_s(mp^r) := \left\{ \boldsymbol{\ell} \in p^s\, \mathbb{Z}^{12}_{\geq 0} \,:\, \boldsymbol{\ell} + \boldsymbol{x} \in U^{(s)}_{\boldsymbol{ab}}(mp^r) \,\text{for some}\, x \in S_{p^s} \right\}.$$

Note that the sets $L_s(mp^r)$ are disjoint as $\boldsymbol{\ell} \in L_s(mp^r)$ implies $\nu_p(\boldsymbol{\ell}) = s$. Finally, consider

$$T_{s,\boldsymbol{\ell}} := \left\{ \boldsymbol{\alpha} \in \mathbb{Z}^{12} \,:\, \boldsymbol{\alpha} = \boldsymbol{\ell} + \boldsymbol{x} \text{ where } \boldsymbol{\ell} \in L_s(mp^r),\, x \in S_{p^s} \right\}.$$

Proposition 2.4. *We have*

$$U_{\boldsymbol{ab}}(mp^r) = \bigsqcup_{1 \leq s \leq r} \; \bigsqcup_{\boldsymbol{\ell} \in L_s(mp^r)} T_{s,\boldsymbol{\ell}}.$$

Proof. We first observe that the sets $T_{s,\boldsymbol{\ell}}$ are disjoint for each fixed s. Let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U_{\boldsymbol{ab}}(mp^r)$. Then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in U^{(s)}_{\boldsymbol{ab}}(mp^r)$ for $s = \min(\nu_p(\boldsymbol{c}), \nu_p(\boldsymbol{d}), r)$. Reducing the fifth and eighth equation in (1.4), we find $p^s \mid b_1$ and $p^s \mid a_3$, respectively. Hence, we may write

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = p^s \boldsymbol{k} + (x, y, 0, 0, z, w, 0, 0, 0, 0, 0, 0)$$

for some $\boldsymbol{k} \in \mathbb{Z}^{12}_{\geq 0}$ and x, y, z, $w \in S_{p^s}$. Now, reducing the first, sixth and seventh equations modulo p^s in (1.4), we obtain $-y \equiv -z \equiv w \equiv x \pmod{p^s}$ and so

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) = p^s \boldsymbol{k'} + \boldsymbol{x} \in U^{(s)}_{\boldsymbol{ab}}(mp^r)$$

where only the components k_2, k_5 and k_6 of $\boldsymbol{k}$ have been redefined to form $\boldsymbol{k'}$. Thus, $\boldsymbol{\ell} := p^s \boldsymbol{k'} \in L_s(mp^r)$ and so $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in T_{s,\boldsymbol{\ell}}$. Conversely, let $\boldsymbol{\alpha} \in T_{s,\boldsymbol{\ell}}$ for some $1 \leq s \leq r$ and $\boldsymbol{\ell} \in L_s(mp^r)$. Then $\boldsymbol{\alpha} = \boldsymbol{\ell} + \boldsymbol{x} = p^s \boldsymbol{k} + \boldsymbol{x} \in U^{(s)}_{\boldsymbol{ab}}(mp^r)$ for some $\boldsymbol{k} \in \mathbb{Z}^{12}_{\geq 0}$ and $x \in S_{p^s}$. Note that, in fact, if we replace the component x in $\boldsymbol{x}$ with any $y \in S_{p^s}$ and write $\boldsymbol{\alpha'}$ for the associated vector, then the eight equations in (1.4) are still satisfied for $\boldsymbol{\alpha'}$. Also, since x, $y \in S_{p^s}$, $p^s k_2 - x \geq 0$ and $p^s k_5 - x \geq 0$, we have $p^s k_2 - y \geq 0$ and $p^s k_5 - y \geq 0$. Hence, $\boldsymbol{\alpha'} \in U^{(s)}_{\boldsymbol{ab}}(mp^r)$ and so $T_{s,\boldsymbol{\ell}} \subseteq U^{(s)}_{\boldsymbol{ab}}(mp^r)$. □

Let $\lfloor\ \rfloor$ and $\lceil\ \rceil$ denote the usual floor and ceiling functions. The next two results follow from splitting the defined product of the binomial coefficient, according to whether the index is divisible by p or not. The first result is [14, Lemma 2.4]. We omit the proof of the second result.

Lemma 2.5. *For primes p, integers m and integers $k \geq 0$, $s \geq 1$,*

$$\binom{mp^s - 1}{k}(-1)^k \equiv \binom{mp^{s-1} - 1}{\lfloor k/p \rfloor}(-1)^{\lfloor k/p \rfloor} \pmod{p^s}.$$

Lemma 2.6. *For primes p, integers m and integers k, $\ell \geq 0$, $s \geq 1$,*

$$\binom{mp^s - \ell}{kp^s} \equiv \binom{mp^{s-1} - \lceil \ell/p \rceil}{kp^{s-1}} \pmod{p^s}.$$

For tuples $\boldsymbol{\ell} := (\ell_1, \ldots, \ell_{12}) \in \mathbb{Z}^{12}$, integers n and $x \in \mathbb{R}$, define

$$C(\boldsymbol{\ell}, n, x) = \binom{n-1}{\ell_1 + \lfloor x \rfloor}\binom{n-\ell_1-\lceil x \rceil}{\ell_3}\binom{n-1}{\ell_6 - \lfloor x \rfloor}\binom{n-\ell_6-\lceil x \rceil}{\ell_4}. \tag{2.8}$$

The reason for the choice of (2.8) will be made clear in the proof of Theorem 1.2, in particular see (3.12).

Lemma 2.7. *Let p be a prime such that p^s divides both $\boldsymbol{\ell}$ and n for some $s \geq 1$ and $x \in \mathbb{R}$ with $0 \leq x \leq p^s$. Then*

$$C(\boldsymbol{\ell}, n, x) \equiv C(\boldsymbol{\ell}/p, n/p, x/p) \pmod{p^s}.$$

Proof. Let $\boldsymbol{\ell} = (p^s k_1, \ldots, p^s k_{12})$ and $n = mp^s$ where k_i, $m \in \mathbb{Z}$ for $1 \leq i \leq 12$. After applying Lemmas 2.5 and 2.6 and simplifying, we obtain

$$\begin{aligned} C(\boldsymbol{\ell}, mp^s, x) &= \binom{mp^s-1}{p^s k_1 + \lfloor x \rfloor}\binom{mp^s - p^s k_1 - \lceil x \rceil}{p^s k_3}\binom{mp^s-1}{p^s k_6 + \lfloor x \rfloor}\binom{mp^s - p^s k_6 - \lceil x \rceil}{p^s k_4} \\ &\equiv \binom{mp^{s-1}-1}{p^{s-1}k_1 + \lfloor x/p \rfloor}\binom{mp^{s-1} - p^{s-1}k_1 - \lceil x/p \rceil}{p^{s-1}k_3}\binom{mp^{s-1}-1}{p^{s-1}k_6 + \lfloor x/p \rfloor} \\ &\quad \times \binom{mp^{s-1} - p^{s-1}k_6 - \lceil x/p \rceil}{p^{s-1}k_4} \pmod{p^s} \\ &\equiv C(\boldsymbol{\ell}/p, mp^{s-1}, x/p) \pmod{p^s}. \end{aligned}$$

□

3. Proof of Theorem 1.2

Proof of Theorem 1.2. We begin by splitting the sum in (1.3) as

$$\begin{aligned} A_\delta(mp^r) &= \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \\ &= \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r) \\ p \mid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) + \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r) \\ p \nmid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}). \end{aligned} \tag{3.1}$$

By Proposition 2.3 and the fact that $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^{r-1})$ if and only if $p(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r)$,

$$\begin{aligned} \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r) \\ p \mid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) &\equiv \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r) \\ p \mid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B((\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})/p) \pmod{p^{3r}} \\ &\equiv \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^{r-1})} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \pmod{p^{3r}} \\ &\equiv A_\delta(mp^{r-1}) \pmod{p^{3r}}. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), it suffices to prove

$$\sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \in U(mp^r) \\ p \nmid (\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \equiv 0 \pmod{p^{3r}}. \tag{3.3}$$

Note that if p does not divide one of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ or $\boldsymbol{d}$, then there must be at least one associated component which is not divisible by p. Choose one of the last four equations in (1.4) such that this component appears and reduce it modulo p. One then finds that p also does not divide at least one of the other two components in this equation. Thus, p does not divide at least two of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ and so the left-hand side of (3.3) becomes

$$\begin{aligned}\sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) &= \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \text{ and } \boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) + \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid 3 \text{ of } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \\ &\quad + \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid 2 \text{ of } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}).\end{aligned} \tag{3.4}$$

We first claim that

$$\sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \text{ and } \boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \equiv \sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid 3 \text{ of } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \equiv 0 \pmod{p^{3r}}. \tag{3.5}$$

To see (3.5), we observe that if p does not divide $\boldsymbol{m} \in \{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\}$, then p does not divide an associated component, say, m_i. So,

$$\binom{mp^r}{\boldsymbol{m}} = \binom{mp^r}{m_i}\binom{mp^r - m_i}{m_j} \tag{3.6}$$

where $j \neq i$. Then by (2.1), the right-hand side of (3.6) is divisible by p^r. Hence, if p does not divide $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ or p does not divide three of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, then p^{3r} divides $B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ and thus (3.5) follows.

Now, consider the set $U_{\boldsymbol{ab}}(mp^r)$ given by (2.7). The sets $U_{\boldsymbol{ac}}(mp^r)$, $U_{\boldsymbol{ad}}(mp^r)$, $U_{\boldsymbol{bc}}(mp^r)$, $U_{\boldsymbol{bd}}(mp^r)$ and $U_{\boldsymbol{cd}}(mp^r)$ are similarly defined. Then,

$$\begin{aligned}\sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid 2 \text{ of } \boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) &= \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{ab}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) + \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{ac}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \\ &\quad + \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{ad}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) + \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{bc}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \\ &\quad + \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{bd}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) + \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{cd}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}).\end{aligned} \tag{3.7}$$

Our second claim is that each of the sums on the right-hand side of (3.7) are equal. To deduce this, consider the maps on $U(mp^r)$ given by

$$(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3,d_1,d_2,d_3) \mapsto (d_2,d_3,d_1,a_2,a_1,a_3,c_2,c_3,c_1,b_3,b_2,b_1), \tag{3.8}$$

$$(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3,d_1,d_2,d_3) \mapsto (b_2,b_1,b_3,d_3,d_2,d_1,c_3,c_1,c_2,a_3,a_1,a_2), \tag{3.9}$$

$$(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3,d_1,d_2,d_3) \mapsto (a_3,a_2,a_1,c_1,c_3,c_2,b_1,b_3,b_2,d_1,d_3,d_2). \tag{3.10}$$

Applying (3.8) or (3.9) does not change the sign or the product of the multinomial coefficients in (2.2) and so if $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) \mapsto (\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, then $B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) = B(\boldsymbol{e},\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. The map (3.8) yields bijections between $U_{\boldsymbol{ab}}(mp^r)$ and $U_{\boldsymbol{ad}}(mp^r)$ and $U_{\boldsymbol{ac}}(mp^r)$ and $U_{\boldsymbol{cd}}(mp^r)$. Similarly, the map

(3.9) gives bijections between $U_{\boldsymbol{ad}}(mp^r)$ and $U_{\boldsymbol{bd}}(mp^r)$ and $U_{\boldsymbol{cd}}(mp^r)$ and $U_{\boldsymbol{bc}}(mp^r)$. Applying (3.10), one can check that (2.2) is unchanged and we have a bijection between $U_{\boldsymbol{ab}}(mp^r)$ and $U_{\boldsymbol{ac}}(mp^r)$. Thus, the second claim follows and so by (3.4), (3.5) and (3.7), we have

$$\sum_{\substack{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U(mp^r)\\ p\nmid(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) = 6\left(\sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{ab}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\right).$$

By Proposition 2.4,

$$\sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in U_{\boldsymbol{ab}}(mp^r)} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) = \sum_{1\le s\le r}\ \sum_{\boldsymbol{\ell}\in L_s(mp^r)}\ \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in T_{s,\boldsymbol{\ell}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}). \tag{3.11}$$

Focusing on the inner sum in (3.11) yields

$$\begin{aligned}
\sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in T_{s,\boldsymbol{\ell}}} B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) &= \sum_{(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\in T_{s,\boldsymbol{\ell}}} (-1)^{a_2+b_1+d_3}\binom{mp^r}{\boldsymbol{a}}\binom{mp^r}{\boldsymbol{b}}\binom{mp^r}{\boldsymbol{c}}\binom{mp^r}{\boldsymbol{d}}\\
&= \sum_{x=1}^{p^s-1}{}' (-1)^{p^sk_2-x+p^sk_4+p^sk_{12}}\binom{mp^r}{\boldsymbol{a}}\binom{mp^r}{\boldsymbol{b}}\binom{mp^r}{\boldsymbol{c}}\binom{mp^r}{\boldsymbol{d}}\\
&= (-1)^{p^sk_2+p^sk_4+p^sk_{12}}\binom{mp^r}{\boldsymbol{c}}\binom{mp^r}{\boldsymbol{d}}\sum_{x=1}^{p^s-1}{}'(-1)^x\binom{mp^r}{\boldsymbol{a}}\binom{mp^r}{\boldsymbol{b}}\\
&= (-1)^{p^sk_2+p^sk_4+p^sk_{12}}m^2p^{2r}\binom{mp^r}{\boldsymbol{c}}\binom{mp^r}{\boldsymbol{d}}\\
&\quad\times\sum_{x=1}^{p^s-1}{}'\frac{(-1)^x}{a_1b_3}\binom{mp^r-1}{a_1-1}\binom{mp^r-a_1}{a_3}\binom{mp^r-1}{b_3-1}\binom{mp^r-b_3}{b_1}
\end{aligned} \tag{3.12}$$

where $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}) = p^s\boldsymbol{k}+\boldsymbol{x}$ and the last step in (3.12) follows from two instances of (2.1).

Now, since $s = \min(\nu_p(\boldsymbol{c}),\nu_p(\boldsymbol{d}),r)$, we find $p^{r-s} \mid \binom{mp^r}{\boldsymbol{c}}\binom{mp^r}{\boldsymbol{d}}$. Hence by (3.12) it suffices to show that

$$\begin{aligned}
&\sum_{x=1}^{p^s-1}{}'\frac{(-1)^x}{a_1b_3}\binom{mp^r-1}{a_1-1}\binom{mp^r-a_1}{a_3}\binom{mp^r-1}{b_3-1}\binom{mp^r-b_3}{b_1}\\
&\equiv \sum_{x=1}^{p^s-1}{}'\frac{(-1)^x}{(p^sk_1+x)(p^sk_6+x)}\binom{mp^r-1}{p^sk_1+x-1}\binom{mp^r-p^sk_1-x}{p^sk_3}\binom{mp^r-1}{p^sk_6+x-1}\\
&\qquad\qquad\times\binom{mp^r-p^sk_6-x}{p^sk_4}\pmod{p^s}\\
&\equiv \sum_{x=1}^{p^s-1}{}'\frac{(-1)^x}{x^2}\binom{mp^r-1}{p^sk_1+x-1}\binom{mp^r-p^sk_1-x}{p^sk_3}\binom{mp^r-1}{p^sk_6+x-1}\binom{mp^r-p^sk_6-x}{p^sk_4}\pmod{p^s}\\
&\equiv 0 \pmod{p^s}.
\end{aligned} \tag{3.13}$$

If we now apply Lemmas 2.5 and 2.6 to (3.13), then it suffices to show

$$\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p, mp^{r-1}, x/p) \equiv 0 \pmod{p^s}. \tag{3.14}$$

To deduce (3.14), our final claim is that

$$\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p, mp^{r-1}, x/p) \equiv \sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t) \pmod{p^s} \tag{3.15}$$

for each $1 \leq t \leq s$. The case $t = 1$ clearly holds. Suppose $t < s$. If $\lfloor x/p^t \rfloor = \lfloor y/p^t \rfloor$ for some $x, y \in S_{p^s}$, then $\lceil x/p^t \rceil = \lceil y/p^t \rceil$ since $x/p^t, y/p^t \notin \mathbb{Z}$ and so $C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t) = C(\boldsymbol{\ell}/p^t, mp^{r-t}, y/p^t)$. Hence if $\lfloor x/p^t \rfloor$ is fixed, $C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t)$ is constant. Also, each $x \in S_{p^s}$ can be written as $x = np^t + y$ where $0 \leq n \leq p^{s-t} - 1$ and $y \in S_{p^t}$. Thus, by Lemma 2.7 with s replaced by $s - t$ and the fact that

$$\sum_{x=np^t+1}^{np^t+p^t-1}{}' \frac{(-1)^x}{x^2} \equiv \sum_{x=1}^{p^t-1}{}' \frac{(-1)^x}{x^2} \equiv 0 \pmod{p^t}$$

which follows from Lemma 2.2 with s replaced by t, we have

$$\begin{aligned}
\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p, mp^{r-1}, x/p) &\equiv \sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t) \pmod{p^s} \\
&\equiv \sum_{n=1}^{p^{s-t}-1}{}' \left(\sum_{x=np^t+1}^{np^t+p^t-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t) \right) \pmod{p^s} \\
&\equiv \sum_{n=1}^{p^{s-t}-1} C(\boldsymbol{\ell}/p^t, mp^{r-t}, x/p^t) \left(\sum_{x=np^t+1}^{np^t+p^t-1}{}' \frac{(-1)^x}{x^2} \right) \pmod{p^s} \\
&\equiv \sum_{n=1}^{p^{s-t}-1} C(\boldsymbol{\ell}/p^{t+1}, mp^{r-t-1}, x/p^{t+1}) \left(\sum_{x=np^t+1}^{np^t+p^t-1}{}' \frac{(-1)^x}{x^2} \right) \pmod{p^s} \\
&\equiv \sum_{n=1}^{p^{s-t}-1} \sum_{x=np^t+1}^{np^t+p^t-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^{t+1}, mp^{r-t-1}, x/p^{t+1}) \pmod{p^s} \\
&\equiv \sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^{t+1}, mp^{r-t-1}, x/p^{t+1}) \pmod{p^s}.
\end{aligned}$$

Thus, (3.15) follows by induction on t. If we now let $t = s$ in (3.15), use the fact that if $\lfloor x/p^s \rfloor$ is fixed, $C(\boldsymbol{\ell}/p^s, mp^{r-s}, x/p^s)$ is constant and apply Lemma 2.2, then

$$\begin{aligned}\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p, mp^{r-1}, x/p) &\equiv \sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} C(\boldsymbol{\ell}/p^s, mp^{r-s}, x/p^s) \pmod{p^s} \\ &\equiv C(\boldsymbol{\ell}/p^s, mp^{r-s}, x/p^s) \left(\sum_{x=1}^{p^s-1}{}' \frac{(-1)^x}{x^2} \right) \pmod{p^s} \\ &\equiv 0 \pmod{p^s}.\end{aligned}$$

This proves (3.14) and thus (1.2). □

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School of Mathematics and Statistics, University College Dublin, Belfield, Dublin 4, Ireland

Theoretical Sciences Visiting Program, Okinawa Institute of Science and Technology Graduate University, Onna, 904-0495, Japan
Email address: robert.osburn@ucd.ie
Email address: brendan.alinquant@ucdconnect.ie