Boulder Transport with Mixed-Lubrication Friction

Jessica DuBerry-Mahon, James Herterich
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We examine the dynamics of boulder transport as a boulder is struck by a storm wave and slides along a platform. Our main point of interest is the friction model used; we replace a typical Coulomb model with a mixed-lubrication model and investigate how the distance moved by the boulder changes. We employ the use of dimensional analysis, examine the effects of changing certain parameters under the mixed-lubrication model, and compare our Coulomb and mixed-lubrication models to a third model - the Imamura model - as further study.

I. INTRODUCTION

Boulder transport by storm waves is a relatively new area of study; before 2014, it was uncertain whether storm waves had the power to dislodge and move coastal boulders at all [1].

While this uncertainty has since been erased, the dynamics of boulder transport by high-energy storm waves are yet to be comprehensively understood [2]. While many coastal civilisations are located in low-energy regions where boulder transport is well studied, as climate change becomes a more serious problem these regions may become increasingly high-energy, and thus increasingly dangerous.

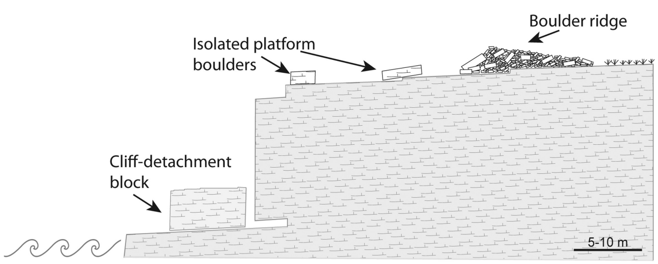

FIG. 1: A profile of coastal platforms [1]. Note the formation of a boulder ridge above the high-tide line.

By increasing our knowledge in this area, we gain an understanding of the level of damage which may be caused by these storm waves, and better prepare for it.

II. THEORY

A. Forces and Equation of Motion

Governing the motion of the boulder, we have our main equation of motion:

$$\rho_s abc \frac{d^2X}{dt^2} = F_d - F_f \tag{1}$$

where:

$$F_d = \frac{1}{2}\rho_s C_d ac \left(V - \frac{dX}{dt}\right)^2 \tag{2a}$$

$$F_l = \frac{1}{2}\rho_s C_l ab \left(V - \frac{dX}{dt}\right)^2 \tag{2b}$$

$$F_g = (\rho_s - \rho_w)gabc \tag{2c}$$

$$F_f = \mu\left(F_g - F_l\right) \tag{2d}$$

Equations 2a, 2b, 2c and 2d represent drag, lift, gravitational and friction forces, respectively. ρ_s and ρ_w are boulder and water density, C_d and C_l are standard hydrodynamic coefficients taken from the literature (1.95 and 0.178, in our case), g is acceleration due to gravity, and a, b and c are the boulder's dimensions. V is fluid velocity, and X is boulder position.

For ease of analysis, we nondimensionalise by taking the boulder length axis b to be our typical x-position scale and V_{max} to be our typical fluid velocity scale, giving us our dimensionless variables:

$$X = b\bar{X} \tag{3a}$$

$$V = V_{max}\bar{V} \tag{3b}$$

$$t = \frac{b}{V_{max}}\bar{t} \tag{3c}$$

Substituting back into equation 1 and dropping the bars, we have:

$$\frac{d^2X}{dt^2} = \frac{1}{2}C_d\left(V - \frac{dX}{dt}\right)^2 - \mu\left(\frac{(1-\rho)}{F^2} - \frac{1}{2}\rho C_l \delta\left(V - \frac{dX}{dt}\right)^2\right)$$

where

$$\rho = \frac{\rho_w}{\rho_s} \tag{4a}$$

$$\delta = \frac{b}{c} \tag{4b}$$

$$F = \frac{V_{max}}{\sqrt{gb}} \tag{4c}$$

are dimensionless numbers for the density ratio, aspect ratio and Froude number, respectively.

The free parameters of aspect ratio and Froude number are of particular interest when examining the mixed-lubrication model.

B. Friction

The focus of our research is on the friction model used in equation 2d, μ. A typical model would be a Coulomb model:

$$\mu = \mu_0$$

where μ_0 is taken to be a constant.

An alternative model is a dynamic friction:

$$\mu = \mu(u)$$

where μ is a function of the boulder's horizontal speed u.

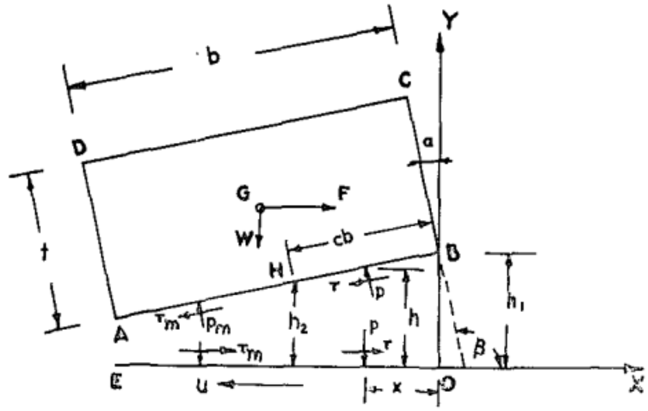

FIG. 2: A cross section of a "boulder" inclined above a plane [3]. An incident water wave would impact from the right. Note changes in notation compared to section II A; boulder height, previously a, is now t, and boulder width, c (as would extend into the page), is now w. In this case, c is taken to be HB/AB, the ratio of the lubricated boulder section to its entire length.

A general formula for the Coulomb friction coefficient is as follows:

$$\mu_0 = \frac{\tau}{p} \tag{5}$$

where τ is shear stress and p is pressure. Generally, taking μ to be constant is overly simplistic. Therefore, we turn to a mixed-lubrication model.

Under this model, we take the boulder to be in solid contact with the platform along discrete regions. In these regions, we take $\mu = \mu_0$. The remaining region is in fully-lubricated contact with the ground, where we take $\mu = \mu(u)$. This is illustrated in Fig. 2, where the region AH is in solid contact and HB is in fully-lubricated contact where the height is of slope α such that

$$\frac{dh}{dx} = \alpha \tag{6}$$

Coulomb friction along the solid-contact region AH is [3]:

$$\mu_0 = \frac{\tau_m}{p_m} \tag{7}$$

where τ_m and p_m are, respectively, average shear stress and pressure in this region, and:

$$p_m = p_0 \left(1 + m g_1 - \bar{p}_1 g_3\right) / g_2 \tag{8}$$

In equation 8, $p_0 = W/bw$ is nominal pressure, $m = \eta u / b p_0$ is Sommerfeld number, $\bar{p}_1 = p_1/p_0$ is nondimensionalised pressure at $h = h_1$, and g_1, g_2 and g_3 are functions taken from the literature [3]. An explicit expression for τ_m is not needed in our case.

In the fully-lubricated region HB, surface shear stresses are [3]:

$$\tau = \frac{\eta u}{h} \pm h \frac{\partial p}{\partial x} \tag{9}$$

where η is dynamic viscosity of the fluid.

The second term in this expression will initially be assumed to be negligible as film thickness h is small. This will be proven in section III D. As such, we have:

$$\tau = \frac{\eta u}{h} \tag{10}$$

Pressure distribution in the region HB is:

$$p = p_1 \phi - \frac{6 u \eta}{\alpha} \left(\frac{1}{h_2} - \frac{1}{h}\right) + \frac{3 u \eta}{\alpha} \left(\frac{h_2 f_2}{f_1}\right) \left(\frac{1}{h_2^2} - \frac{1}{h^2}\right) \tag{11}$$

α is angle of inclination of the boulder, and f_1 and f_2 are functions taken from the literature [3].

We combine a solid contact region and a lubricated region for a mixed lubrication model. The effective friction coefficient may be obtained after a significant derivation (see [3] for details):

$$\mu = \frac{\tau_m b w (1 - c) + (w/\alpha) \int_{h_2}^{h_1} \tau dh}{\mathrm{p}_m b w (1 - c) + (w/\alpha) \int_{h_2}^{h_1} \mathrm{p} dh} \tag{12}$$

Substituting equations 10 and 11 into equation 12, while taking $h = \alpha x + h_1 - \alpha b$, this expression may be simplified to:

$$\mu = f_{17}/f_{18} \tag{13}$$

where:

$$f_{17} = \mu_0 (1 - c) \bar{p}_m + m \ln a/\alpha \tag{14a}$$

$$f_{18} = (1 - c) \bar{p}_m + (1/\alpha) \left[\bar{p}_1 \bar{h}_2 f_6 - m (f_7 - f_8)\right] \tag{14b}$$

$\bar{h}_2 = h_2/h_0$ and f_6, f_7 and f_8 are functions taken from the literature [3].

C. Further Study

For interest, a third dynamic-only friction model is introduced and briefly discussed as a comparison to the

Coulomb and mixed-lubrication model: the Imamura model [4]. In this case, the incident wave is treated as a tsunami wave where it was previously a storm wave.

As such, we have:

$$\mu(u) = \mu_0 \frac{2.2}{\beta^2 + 2.2} \tag{15}$$

where:

$$\beta^2 = \frac{u^2}{(1 - \rho_w/\rho_s)}$$

and μ_0 is the Coulomb coefficient.

In addition, we examine the effects of modelling the boulder as having N separate regions of inclination rather than one. In this case, each region has angle of inclination α_N, where:

$$\alpha_N = \arcsin\left(\frac{N(h_1 - h_2)}{cb}\right) \tag{16}$$

III. RESULTS

Throughout the following section, unless stated otherwise, it may be assumed that $\mu_0 = 0.7$ and $h_2 = 0.001$m.

A. Coulomb and Mixed-Lubrication Friction

Figure 3 shows an example of the evolution of nondimensionalised boulder position and velocity with time, for a Froude number of 2 and aspect ratio of 1 (see equations 4c and 4b). A Coulomb friction model is used in this case. The impact of the storm wave is modelled as a Gaussian pulse which strikes and moves the boulder - in the case of Figure 3, it moves a distance of approximately $0.5b$.

Figure 4 shows the evolution of the mixed-lubrication friction coefficient with nondimensionalised sliding speed. Note that, when the boulder is still, $\mu = \mu_0 = 0.7$. As the speed of the boulder increases, μ decreases. Additionally, as b increases - and with it, δ - μ decreases at a faster rate.

Figure 6 shows a comparison of the distance moved by the boulder under Coulomb and mixed-lubrication friction models. Evidently, distance moved is generally larger under the mixed-lubrication model.

B. Mixed-Lubrication - Changing h_1

Figure 7 shows the distance moved under the mixed-lubrication friction model for changing h_1. Distance and h_1 are inversely proportional - in other words, as angle of inclination α increases, distance moved decreases.

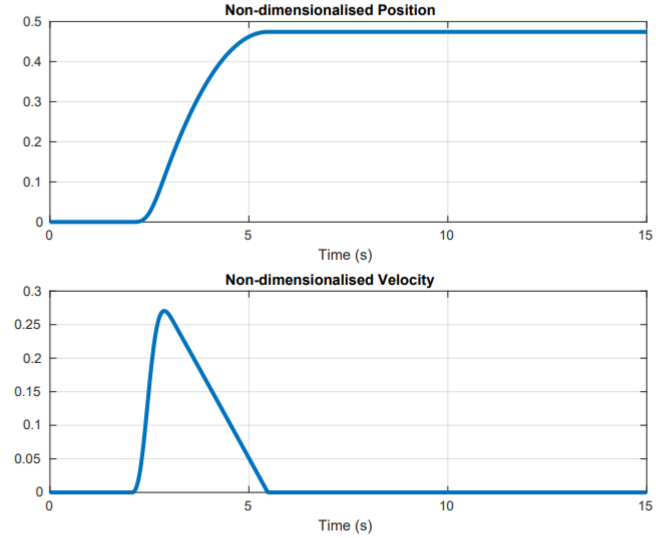

FIG. 3: Evolution of nondimensionalised boulder position and velocity with time.

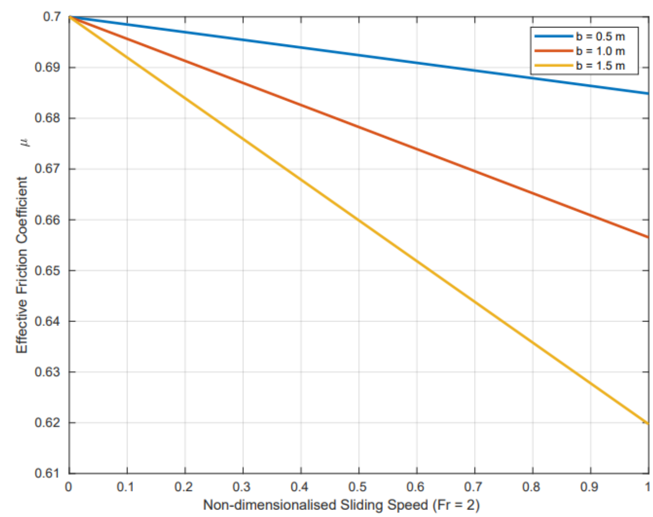

FIG. 4: Evolution of μ with sliding speed, for varying values of boulder length b. Here, $Fr = 2$, $w = 1$m.

C. Mixed-Lubrication - Changing N

Figure 8 shows the distance moved under the mixed-lubrication friction model for changing N. As the number of regions of inclination increases, distance moved decreases.

D. Shear Stress Comparison

Referring to Figure 5, we may prove that the second term in equation 9 is negligible. Clearly, there is essentially no difference to the evolution of μ if the extra term is included.

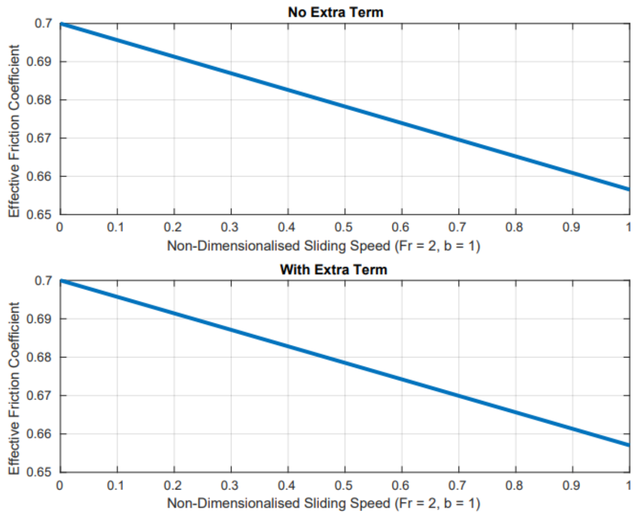

FIG. 5: A comparison of the evolution of μ with nondimensionalised sliding speed, with and without the extra shear stress term.

E. Imamura Model

Finally, Figure 9 shows a comparison of distance moved under Coulomb, mixed-lubrication and Imamura friction models.

Interestingly, distance moved under the Imamura model lies somewhere between the other two.

IV. CONCLUSIONS

We have examined the dynamics of boulder transport by storm waves, focusing on employing a mixed-lubrication friction model that is dependent on boulder sliding speed.

In doing so, we have shown that such a model provides useful insight into how certain parameters, such as the boulder's dimensions or angle of inclination, may affect how far the boulder is moved.

[1] R. Cox, K. L. Jahn, O. G. Watkins, and P. Cox, Extraordinary boulder transport by storm waves (west of Ireland, winter 2013-2014), and criteria for analysing coastal boulder deposits, Earth-Science Reviews **177**, 623 (2018).

[2] R. Cox, F. Ardhuin, F. Dias, R. Autret, N. Beisiegel, C. S. Earlie, J. G. Herterich, A. Kennedy, R. Paris, A. Raby, *et al.*, Systematic review shows that work done by storm waves can be misinterpreted as tsunami-related because commonly used hydrodynamic equations are flawed, Frontiers in Marine Science **7**, 4 (2020).

[3] D. S. Bedi and M. J. Hillier, Mixed Lubrication of "Parallel" Plane Sliders, Journal of Applied Mechanics **1**, 36 (1972).

[4] F. Imamura, K. Goto, and S. Ohkubo, A numerical model for the transport of a boulder by tsunami, Journal of Geophysical Research **113** (2009).

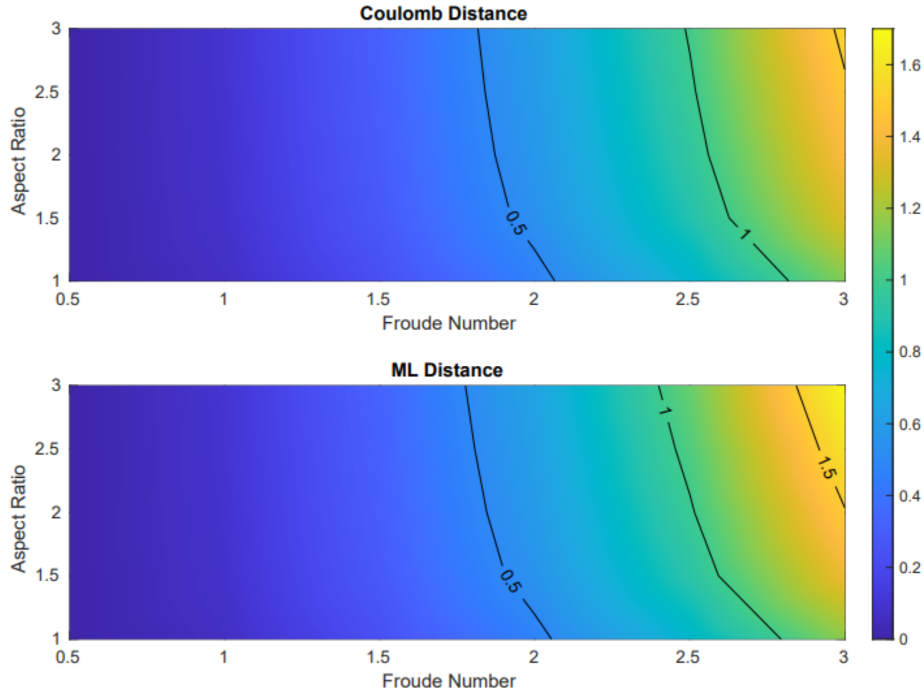

FIG. 6: Nondimensionalised distance moved by the boulder under Coulomb and mixed-lubrication (ML) friction models for various values of Froude number and aspect ratio. Here, $h_1 = 0.005$m.

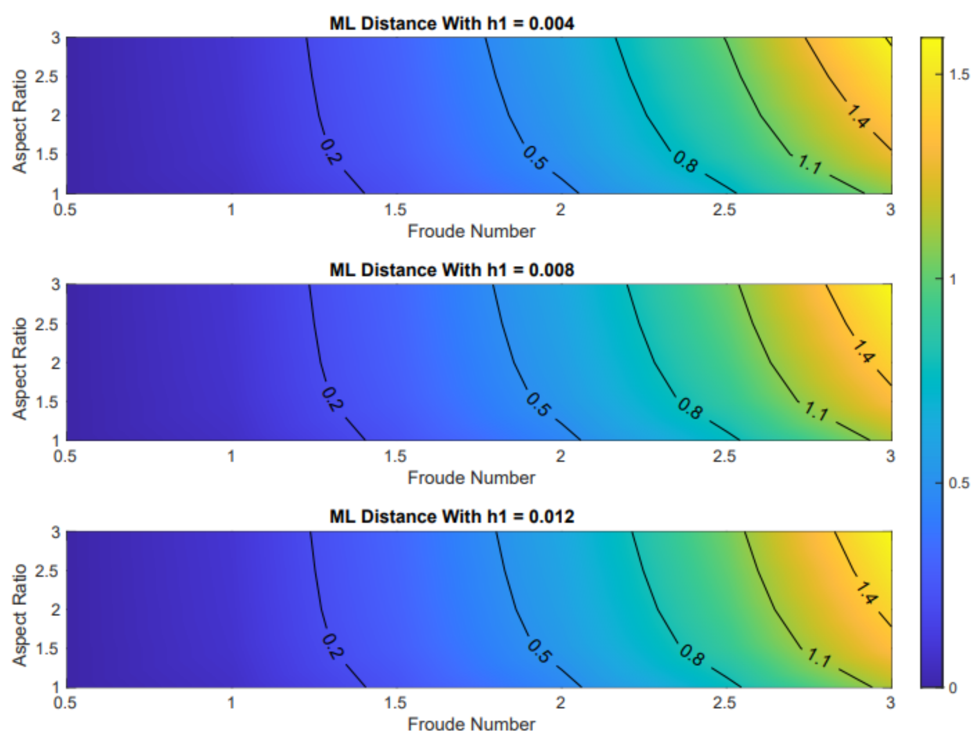

FIG. 7: Nondimensionalised distance moved by the boulder under the mixed-lubrication model for changing h_1.

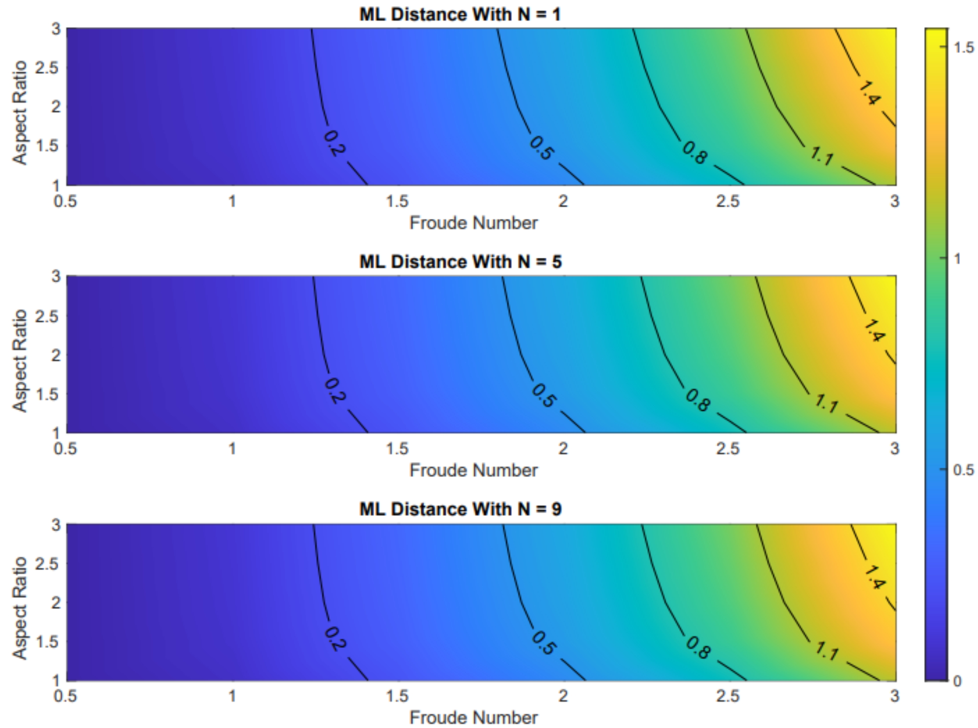

FIG. 8: Nondimensionalised distance moved by the boulder under the mixed-lubrication model for changing N.

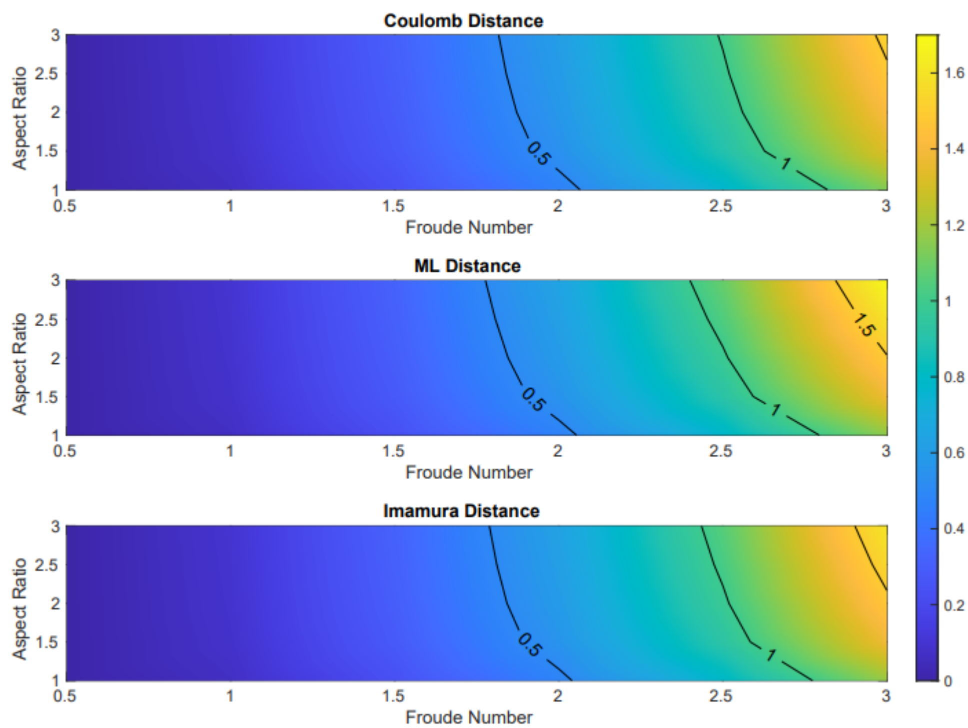

FIG. 9: Nondimensionalised distance moved by the boulder under Coulomb, mixed-lubrication and Imamura friction models. Here, $h_1 = 0.005$m.