The non-negative inverse eigenvalue problem
Undergraduate Summer Research Project Report

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Abstract

The non-negative inverse eigenvalue problem is an open question in matrix theory and regarded as one of the most difficult in linear algebra and matrix theory over the past 50 years. In this report I outline a general solution for matrices of low dimension and discuss the progress made for larger matrices, including some necessary conditions. Several results on the problem are obtained from considering particular classes of matrices.

1 Introduction

Definition 1.1. A complex number λ is an *eigenvalue* of an $n \times n$ matrix A if

$$Av = \lambda v$$

for some $n \times 1$ nonzero vector v. The vector v is said to be an *eigenvector* of A corresponding to the eigenvalue λ.

Definition 1.2. The list of eigenvalues of a matrix A is called its *spectrum*, often denoted $\sigma(A)$ or simply σ.

Definition 1.3. A matrix A is said to be *non-negative*, written $A \geq 0$, if all its entries are non-negative.

The non-negative inverse eigenvalue problem (NIEP) is an open problem in matrix theory. Matrices occur across a multitude of disciplines, including finance and physics. A certain type of non-negative matrix known as a stochastic matrix is used in probability theory and Google's PageRank algorithm. There is particular interest in non-negative matrices as real-world applications typically deal with non-negative numbers. For instance, non-negative matrices can represent financial losses or gains, physical measurements and probabilities.

2 NIEP

In 1937, Kolmogorov [?] asked what is considered to be the precursor to the non-negative inverse eigenvalue problem: When is a given complex number an eigenvalue of a non-negative matrix? This question has since been answered and a solution is described later in this report.

The non-negative inverse eigenvalue problem was first posed by Suleĭmanova in 1949 [?]. It asks for necessary and sufficient conditions such that a list of complex numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the spectrum of a $n \times n$ non-negative matrix A. If there exists such a matrix A then σ is said to be realizable and σ is realized by A. The NIEP remains unsolved today, though advances have been made for small values of n, which I discuss in this section.

2.1 Some necessary conditions

Let A be a non-negative $n \times n$ matrix with spectrum $\sigma = (\lambda_1, \ldots, \lambda_n)$.

A is similar to its Jordan canonical form J, which has the elements of σ along its main diagonal, thus

$$\text{trace}(A) := \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \text{trace}(J)$$

Since $a_{ij} \geq 0 \; \forall \, i, j \in (1, \ldots, n)$, we have that trace$(A) \geq 0$. Hence for any $A \geq 0$,

$$\sum_{i=1}^{n} \lambda_i \geq 0$$

Let λ be an eigenvalue of A, with corresponding eigenvector v. Then $Av = \lambda v$.
Consider A^2:

$$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$$

By induction, if λ is an eigenvalue of A then λ^k is an eigenvalue of $A^k \; \forall \, k \in \{1, 2, \ldots\}$.
If $A \geq 0$, then $A^k \geq 0$ also $\forall \, k \in \{1, 2, \ldots\}$, giving the first necessary condition:

$$s_k := \sum_{i=1}^{n} \lambda_i^k \geq 0 \;\; \forall \, k \in \{1, 2, \ldots\}. \tag{I}$$

The Perron-Frobenius theorem is a significant result in matrix theory which reveals another necessary condition. Perron first characterised real square matrices with positive entries in 1907 [?]. His theorem was later extended to irreducible non-negative matrices A (and spectra σ) by Frobenius in 1912 [?]. It states that there exists an eigenvalue $\rho \in \sigma$ such that $\rho \geq |\lambda_i| \; \forall \, i \in (1, \ldots, n)$. The real number ρ is known as the Perron eigenvalue or Perron root. Thus we require

$$\rho = \max_i \{|\lambda_i| : \lambda_i \in \sigma(A)\} \in \sigma(A). \tag{II}$$

Since the entries of A are real, its characteristic polynomial $f(x) = \det(xI - A)$ has real coefficients. Any complex roots of $f(x)$ must thus occur in conjugate pairs by the complex conjugate root theorem. As the roots of $f(x)$ are the eigenvalues of A, we have

$$\sigma = \overline{\sigma}. \tag{III}$$

A fourth necessary condition (JLL condition) was obtained by Loewy and London [?] in 1978 using Hölder's inequality, and independently by Johnson in 1981 [?], which states that

$$n^{m-1} s_{km} \geq s_k^m \;\; \forall \, k, m \in (1, 2, \ldots). \tag{IV}$$

These necessary conditions are not exhaustive. One such additional necessary condition was found by Cronin and Laffey in 2012 [?]. As the four conditions above are necessary but not sufficient, we can use them only to disprove the realizability of spectra or to check the potential of candidate spectra as realizable lists. If a spectrum satisfies the conditions, it may be realizable, but a conclusion cannot be drawn from this alone.

Example 2.1.

1. $\sigma = (1, 0, -2)$ is not realizable since $s_1 = -1 < 0$, and so (I) is failed.

2. $\sigma = (1+i, 1-i)$ is not realizable since $\rho = \max_i(|\lambda_i|) = \sqrt{2} \notin \sigma$, so (II) is betrayed.

3. $\sigma = (1, 0, i)$ is not realizable since $\overline{\sigma} = (1, 0, -i) \neq \sigma$, and so (III) breaks down.

4. $\sigma = (\sqrt{2}, i, -i)$ satisfies the first three conditions, however it is not realizable since it fails the JLL condition with $n = 3, k = 1, m = 2$:

$$3s_2 = 0 < 2 = s_1^2.$$

Definition 2.1. A *permutation matrix* P is a square matrix with exactly one entry equal to 1 in each row and each column, and all other entries equal to 0. It permutes the rows of another matrix when multiplied on the left and the columns of another matrix when multiplied on the right, e.g. if $\pi = (132)$ then PA results in a permutation of A in which row 1 has been sent to row 3, row 3 to 2 and row 2 to 1, where P is the permutation matrix associated with π.

Definition 2.2. Two matrices A and B are *permutation-similar* if $\exists$ permutation matrix P with

$$P^{-1}AP = B \iff P^T AP = B$$

Definition 2.3. A $n \times n$ matrix R is *reducible* if

(i) when $n = 1$, R is the zero matrix

(ii) when $n \geq 2$, $\exists$ permutation matrix P with

$$P^T RP = \begin{bmatrix} R_1 & C \\ 0 & R_2 \end{bmatrix}$$

where R_1 is $(n-r) \times (n-r)$, R_2 is $r \times r$, C is $(n-r) \times r$, 0 is $r \times (n-r)$.

Otherwise, R is an *irreducible* matrix.

The following statement regarding irreducible non-negative matrices is one of several results from the Perron-Frobenius theorem.

Theorem 2.1 ([?],[?]). *If A is a non-negative and irreducible matrix then its Perron root ρ is algebraically (and hence geometrically) simple, i.e. ρ occurs in $\sigma(A)$ once.*

Example 2.2.

1. $\sigma = (\sqrt{2}, \sqrt{2}, i, -i)$ satisfies all four necessary conditions above. However, $\rho = \sqrt{2}$ occurs in σ twice. By Theorem 2.1, any realizing matrix is reducible, comprising two blocks both containing ρ (condition II). Now i and $-i$ must be grouped together by condition III, so the larger block is not realizable by Example 2.1 (4). Hence σ is not realizable and the four necessary conditions are not sufficient conditions.

2. Let $\sigma = (3, 3, -2, -2, -2)$. The Perron root $\rho = 3$ occurs in σ twice, so any realizing matrix R is reducible. Thus R can have one 1×1 block and one 4×4 block, or one 2×2 block and one 3×3 block. The spectrum of the 4×4 block in the first case is not realizable since $s_1 = 3 - 2 - 2 - 2 = -3 < 0$. Similarly, the spectrum of the 3×3 block in the second case is not realizable since $s_1 = 3 - 2 - 2 = -1 < 0$. Hence σ is not realizable.

The spectrum $\sigma_t = (3+t, 3-t, -2, -2, -2)$ is a classical example in the theory of the NIEP. We showed that σ_0 is not realizable in Example 2.2 (2). Cronin and Laffey [?] recently used σ_t to demonstrate that the problem of finding a non-negative diagonalizable matrix that realizes a given real spectrum (D-RNIEP) and the problem of finding a symmetric non-negative matrix that realizes a given spectrum (SNIEP, see section 4.1) are not equivalent. They additionally showed that the smallest $t > 0$ for which σ_t is realizable by a diagonalizable matrix is $t = 1$.

2.2 The case n = 2

When $n = 1$, the NIEP is trivial. The spectrum (λ) is realized by the matrix $\begin{bmatrix}\lambda\end{bmatrix}$, which is non-negative when $\lambda \geq 0$. We consider the $n = 2$ case.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where $a, b, c, d \geq 0$

Let λ_1 and λ_2 be the eigenvalues of A. They are the roots of the characteristic polynomial $f(x) = \det(xI_2 - A)$. Thus we have

$$\det(xI_2 - A) = x^2 - (a+d)x + ad - bc, \text{and}$$
$$(x - \lambda_1)(x - \lambda_2) = x^2 - (\lambda_1 + \lambda_2) + \lambda_1\lambda_2.$$

Comparing coefficients we obtain

$$\lambda_1 + \lambda_2 = a + d \geq 0 \tag{1}$$

$$\lambda_1\lambda_2 = ad - bc. \tag{2}$$

The discriminant of $f(x)$

$$\Delta f(x) = (a-d)^2 + 4bc \geq 0 \implies \lambda_1, \lambda_2 \in \mathbb{R}.$$

Subbing (1) into (2) yields

$$-bc = (a - \lambda_1)(a - \lambda_2). \tag{3}$$

Hence

$$b = (a - \lambda_2)$$

and

$$c = -(a - \lambda_1) = (\lambda_1 - a)$$

Thus

$$M = \begin{bmatrix} a & a - \lambda_2 \\ \lambda_1 - a & \lambda_1 + \lambda_2 - a \end{bmatrix}$$

realizes $\sigma = (\lambda_1, \lambda_2)$, where $0 \leq a \leq \lambda_1 + \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.

Observation 2.1.

(i) $M' = \begin{bmatrix} \lambda_1 + \lambda_2 - a & a - \lambda_2 \\ \lambda_1 - a & a \end{bmatrix}$, M'^T, M^T realize σ also.

(ii) The requirement for two real eigenvalues follows from conditions (II) and (III). The Perron root must be real, so we cannot have a complex conjugate pair.

(iii) From equation (1) and the fact that both eigenvalues are real, we see that condition (I) is satisfied.

(iv) We have shown that sufficient conditions for the $n = 2$ case are simply that $s_1 \geq 0$ and that σ is real.

2.3 The case n = 3

Let $\sigma = (\lambda_1, \lambda_2, \lambda_3)$ be the spectrum of a 3×3 non-negative matrix B.
Let

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

where $a, b, c, d, e, f, g, h, i \geq 0$.

As in the $n = 2$ case, $\lambda_1, \lambda_2, \lambda_3$ are the roots of

$$\begin{aligned} \det(xI - B) &= x^3 + (-a - e - i)x^2 + (ae + ai - bd - cg + ei - fh)x - aei + afh + bdi - bfg - cdh + ceg \\ &= x^3 + (-\lambda_1 - \lambda_2 - \lambda_3)x^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)x + \lambda_1\lambda_2\lambda_3 \end{aligned}$$

Hence

$$\lambda_1 + \lambda_2 + \lambda_3 = a + e + i \tag{4}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = ae + ai - bd - cg + ei - fh \tag{5}$$

$$\lambda_1\lambda_2\lambda_3 = aei + bfg + cdh - ceg - bdi - afh \tag{6}$$

If $\lambda_1, \lambda_2, \lambda_3 \geq 0$ then

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

realizes σ.
In general, if $\lambda_1, \ldots, \lambda_n$ are the non-negative eigenvalues of a matrix then the matrix can be written as the non-negative matrix $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

If $\lambda_1, \lambda_2 \geq 0$ and $\lambda_3 \leq 0$ then

$$\begin{bmatrix} \lambda_1 + \lambda_3 & 0 & \lambda_1 \\ 0 & \lambda_2 & 0 \\ -\lambda_3 & 0 & 0 \end{bmatrix}$$

realizes σ when $\lambda_1 \geq |\lambda_3|$

If $\lambda_1 \geq 0$ and $\lambda_2, \lambda_3 \leq 0$ then

$$\begin{bmatrix} 0 & \lambda_1\lambda_2\lambda_3 & 0 \\ 0 & \lambda_1 + \lambda_2 + \lambda_3 & 1 \\ 1 & -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) & 0 \end{bmatrix}$$

realizes σ when $\lambda_1 \geq |\lambda_2 + \lambda_3|$.

A general solution for 3×3 matrices was provided by Loewy and London in 1978 [?].

They showed that if $\sigma = (\rho, re^{i\theta}, re^{-i\theta})$, where $0 < r \leq \rho$ and $0 < \theta < \pi$, satisfies the four conditions outlined in section 2.1 then

$$\frac{1}{3}\begin{bmatrix} \rho + 2r\cos\theta & \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) \\ \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2r\cos\theta & \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) \\ \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2r\cos\theta \end{bmatrix}$$

realizes σ.

This realizing matrix is an example of a circulant matrix; row i is the first row following a cyclic permutation which shifts $i-1$ places to the right $\forall\, i \in \{1, 2, 3\}$.

2.4 An answer for Kolmogorov

Kolmogorov's question – when is a given complex number z an eigenvalue of a non-negative matrix A? – can now be answered.

If $0 \leq z \in \mathbb{R}$ then $A = \begin{bmatrix} z \end{bmatrix}$ trivially realizes z.

If $0 > z \in \mathbb{R}$ then

$$A = \begin{bmatrix} 0 & -z \\ -z & 0 \end{bmatrix}$$

has eigenvalues $-z$ and z by the $n = 2$ case solution in section 2.2.

If $z = re^{i\theta} \in \mathbb{C} \setminus \mathbb{R}$ then, using Loewy and London's solution from section 2.3,

$$A = \frac{1}{3}\begin{bmatrix} \rho + 2r\cos\theta & \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) \\ \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2r\cos\theta & \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) \\ \rho - 2r\cos\left(\frac{\pi}{3}+\theta\right) & \rho - 2r\cos\left(\frac{\pi}{3}-\theta\right) & \rho + 2r\cos\theta \end{bmatrix}$$

has spectrum $(\rho, re^{i\theta}, re^{-i\theta})$.

If we set $a_{13} = 0$, then $\rho = 2r\cos\left(\frac{\pi}{3} - \theta\right)$, so we can rewrite the matrix A in terms of our chosen complex number $z = re^{i\theta}$.

Now

$$A' = \frac{2r}{3}\begin{bmatrix} \cos\theta + \sin\left(\theta + \frac{\pi}{6}\right) & \sqrt{3}\sin\theta & 0 \\ 0 & \cos\theta + \sin\left(\theta + \frac{\pi}{6}\right) & \sqrt{3}\sin\theta \\ \sqrt{3}\sin\theta & 0 & \cos\theta + \sin\left(\theta + \frac{\pi}{6}\right) \end{bmatrix}$$

has spectrum $(2r\cos\left(\frac{\pi}{3} - \theta\right), re^{i\theta}, re^{-i\theta})$ and is nonnegative for $0 < \theta < \frac{\pi}{4}$.

We have shown that for every complex number z there exists always a non-negative matrix A for which z is an eigenvalue of A.

2.5 Larger matrices

The NIEP was solved for 4×4 matrices with zero trace by Reams [?] in 1996. Meehan [?] solved the general $n = 4$ case using the k-th moments s_k in 1998, while in 2007 Torre-Mayo et al. [?] offered an alternative solution in terms of the coefficients of the characteristic polynomial. When $n = 5$, a solution for matrices with trace zero was constructed by Laffey and Meehan in 1999 [?] using graph cycles and Newton's identities.

3 More results on realizability

Though the NIEP has yet to be solved for matrices of arbitrary size, there are many theorems that can tell us about the realizability of some types of spectra. One such theorem was given by Suleimanova in the paper in which she first posed the NIEP.

Theorem 3.1 (Suleimanova (1949) [?]). *Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be a list of real eigenvalues, with $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then σ is realizable if and only if $s_1 = \sum_{i=1}^{n} \lambda_i \geq 0$.*

A spectrum $(\lambda_1, \ldots, \lambda_n) \subset \mathbb{R}$ that satisfies $\lambda_1 > 0 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $s_1 \geq 0$ is known as a *Suleimanova spectrum*. A refined proof of Suleimanova's theorem was provided by Perfect in 1953 [?]. Indeed, the theorem evidently holds in the $n = 2, 3$ cases given in sections 2.2 and 2.3.

3.1 Insights from special matrices

Definition 3.1. Let $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$, where $a_1, \ldots, a_n \in \mathbb{R}, n \in \mathbb{N}$
Then

$$C(f) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

is the companion matrix of $f(x)$.

Companion matrices are useful for solving non-negative inverse eigenvalue problems because the roots of $f(x)$ are exactly the eigenvalues of $C(f)$. We see that if the non-leading coefficients of $f(x)$ are non-positive, then the roots of $f(x)$ can be realized by $C(f)$.

Friedland [?] provided a further proof of Theorem 3.1 using companion matrices. He showed that Suleimanova spectra are realizable via companion matrices

Example 3.1.

1. Let $\sigma = (2, 1+i, 1-i, 0)$.
   Here $s_1 = 4$. Subtract 1 from each element of σ to obtain $\sigma' = (1, i, -i, -1)$.

   Let $f(x) = (x-1)(x-i)(x+i)(x+1) = x^4 - 1$.
   Then

   $$C(f) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

   is the companion matrix for $f(x)$ which realizes σ'.

   Then

   $$C(f) + I_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

   realizes $\sigma = (2, 1+i, 1-i, 0)$.

2. Let $\sigma = (8, -2, -2, -2, -2)$.
   Here $s_1 = 0$, hence the trace of any realizing matrix is zero.

   Let $f(x) = (x-8)(x+2)^4 = x^5 - 40x^3 - 160x^2 - 240x - 128$.
   Then

$$D(f) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 128 & 240 & 160 & 40 & 0 \end{bmatrix}$$

   is the companion matrix of $f(x)$ and realizes σ.

Definition 3.2. A square matrix S is symmetric if $S = S^T$.

Definition 3.3. A square matrix Q is orthogonal if $QQ^T = Q^TQ = I$, i.e. $Q^T = Q^{-1}$. The columns (and rows) of Q are orthogonal unit vectors.

Definition 3.4. Let R be an $n \times n$ real orthogonal matrix with columns $r_1, \ldots, r_n$. R is called a Soules matrix if r_1 is positive and for every $D := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the matrix RDR^T is non-negative.

Example 3.2. First we construct a matrix such that the columns are pairwise orthogonal:

$$\begin{bmatrix} 1 & \_ & \_ & \_ & \_ \\ 1 & \_ & \_ & \_ & \_ \\ 1 & \_ & \_ & \_ & \_ \\ 1 & \_ & \_ & \_ & \_ \\ 1 & \_ & \_ & \_ & \_ \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & \_ & \_ & \_ \\ 1 & 1 & \_ & \_ & \_ \\ 1 & 1 & \_ & \_ & \_ \\ 1 & 1 & \_ & \_ & \_ \\ 1 & -4 & \_ & \_ & \_ \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & \_ & \_ \\ 1 & 1 & 1 & \_ & \_ \\ 1 & 1 & 1 & \_ & \_ \\ 1 & 1 & -3 & \_ & \_ \\ 1 & -4 & 0 & \_ & \_ \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -2 & 0 \\ 1 & 1 & -3 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 \end{bmatrix}$$

Next we normalize each column to obtain a 5×5 Soules matrix:

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{20}} & \frac{1}{\sqrt{12}} & \frac{-\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{20}} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} & 0 & 0 & 0 \end{bmatrix}$$

We can verify that R is a Soules matrix. Let $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_5)$, where $\lambda_1 \geq \cdots \geq \lambda_5 \geq 0$. Then $RDR^T =$

$$\frac{1}{5}\begin{bmatrix} \frac{12\lambda_1+3\lambda_2+5\lambda_3+10\lambda_4+30\lambda_5}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3+10\lambda_4-30\lambda_5}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3-20\lambda_4}{12} & \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \lambda_1-\lambda_2 \\ \frac{12\lambda_1+3\lambda_2+5\lambda_3+10\lambda_4-30\lambda_5}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3+10\lambda_4+30\lambda_5}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3-20\lambda_4}{12} & \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \lambda_1-\lambda_2 \\ \frac{12\lambda_1+3\lambda_2+5\lambda_3-20\lambda_4}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3-20\lambda_4}{12} & \frac{12\lambda_1+3\lambda_2+5\lambda_3+40\lambda_4}{12} & \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \lambda_1-\lambda_2 \\ \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \frac{4\lambda_1+\lambda_2-5\lambda_3}{4} & \frac{4\lambda_1+\lambda_2+15\lambda_3}{4} & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \lambda_1+4\lambda_2 \end{bmatrix} \geq 0$$

Theorem 3.2 (Soules (1983) [?], Elsner, Nabben, Neumann (1998) [?])**.** *Let R be a Soules matrix and let $D := \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then the off-diagonal entries of the matrix RDR^T are non-negative.*

Example 3.3. Let $D = \operatorname{diag}(10, 4, 3, -2, -3)$ and let R be the Soules matrix from Example 3.3. Then

$$RDR^T = \begin{bmatrix} \frac{442}{15} & \frac{217}{15} & \frac{599}{30} & \frac{193}{10} & \frac{84}{5} \\ \frac{217}{15} & \frac{442}{15} & \frac{599}{30} & \frac{193}{10} & \frac{84}{5} \\ \frac{599}{30} & \frac{599}{30} & \frac{719}{30} & \frac{193}{10} & \frac{84}{5} \\ \frac{193}{10} & \frac{193}{10} & \frac{193}{10} & \frac{253}{10} & \frac{84}{5} \\ \frac{84}{5} & \frac{84}{5} & \frac{84}{5} & \frac{84}{5} & \frac{164}{5} \end{bmatrix}$$

is a non-negative matrix having spectrum $\sigma = (10, 4, 3, -2, -3)$.

Note that the non-negative property was not guaranteed. However, the only entries that needed to be checked for their signs were the five diagonal entries. The theorem thus reduces the number of checks required from n^2 to n.

3.2 C-realizability

Observation 3.1. Observe that if $\sigma = (\lambda_1, \ldots, \lambda_m)$ and $\sigma' = (\mu_1, \ldots, \mu_n)$ are realizable spectra, then the list $(\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_n)$ is also realizable.

Proof. If A realizes σ and B realizes σ' then

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

realizes the adjoined spectra. □

Theorem 3.3 (corollary of Brauer's theorem [?])**.** *Let $(\rho, \lambda_2, \ldots, \lambda_n)$ be the spectrum of a non-negative matrix having Perron root ρ. Then $(\rho + \epsilon, \lambda_2, \ldots, \lambda_n)$ is realizable $\forall\, \epsilon \geq 0$.*

Theorem 3.4 (Guo [?] (1997))**.** *Let $(\rho, \lambda_2, \ldots, \lambda_n)$ be the spectrum of a non-negative matrix having Perron root ρ, where $\lambda_2 \in \mathbb{R}$. Then $(\rho + \epsilon, \lambda_2 \pm \epsilon, \ldots, \lambda_n)$ is realizable $\forall\, \epsilon \geq 0$.*

Definition 3.5 ([?])**.** A list of real eigenvalues $(\lambda_1, \ldots, \lambda_n)$ is said to be *C-realizable* if it can be obtained by starting with the n trivially realizable spectra $(0), \ldots, (0)$ and applying Observation 3.1, Theorem 3.4 and Theorem 3.5 any number of times in any order.

Example 3.4.

- Begin with eight copies of the trivially realizable spectrum (0):
  $(0), (0), (0), (0), (0), (0), (0), (0)$
- Apply Obs. 3.1 four times:
  $(0, 0), (0, 0), (0, 0), (0, 0)$
- Apply Thm. 3.5 four times:
  $(4, -4), (3, -3), (6, -6), (2, -2)$
- Apply Obs. 3.1 twice:
  $(4, 3, -3, -4), (6, 2, -2, -6)$
- Apply Thm. 3.4 twice for $\epsilon_1 = 3, \epsilon_2 = 2$:
  $(7, 3, -3, -4), (8, 2, -2, -6)$

- Apply Obs. 3.1 once:
  $(8, 7, 3, 2, -2, -3, -4, -6)$

- Apply Thm. 3.5 once:
  $(9, 7, 3, 2, -2, -3, -5, -6)$

Hence $(9, 7, 3, 2, -2, -3, -5, -6)$ is realizable, though this method does not indicate how to construct a realizing matrix.

Theorems 3.4 and 3.5 give importance to a particular characteristic of spectra.

Definition 3.6. The *spectral gap* of a non-negative matrix is the difference between its Perron eigenvalue ρ and the absolute value of its second-largest eigenvalue.

Given two spectra, a smaller spectral gap is advantageous as it allows a greater number of spectra to be classified as realizable (or otherwise) by repeated use of Theorems 3.4 and 3.5. For instance, $(8, -2, -2, -2, -2)$ is realizable by Example 3.2 (2), thus $(8 + \epsilon, -2 \pm \epsilon, -2, -2, -2)$ is also realizable $\forall \epsilon \geq 0$. However, if we had started with $(100, -2, -2, -2, -2)$ being realizable, we could not easily comment on the realizability of $(8, -2, -2, -2, -2), (9, -2, -2, -2, -2), \ldots, (99, -2, -2, -2, -2)$ using Theorem 3.5.

3.3 Boyle-Handelman theorem

The following theorem is a highly celebrated result first published in the Annals of Mathematics.

Theorem 3.5 (Boyle, Handelman [?] (1991)). *If*

(i) σ *has a Perron element* $\lambda_1 > |\lambda_j| \ \forall j > 1$ *and*

(ii) $s_k \geq 0 \ \forall$ *positive integers* k *(and* $s_m = 0$ *for some* m *implies* $s_d = 0$ *for all positive divisors* d *of* m*)*

then $\sigma_N := (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$ *(*N *zeros) is realizable for all sufficiently large* N*.*

The Boyle-Handelman theorem can be used only to prove the existence of a realizing matrix. Furthermore, the proof of the theorem requires knowledge beyond the usual scope of matrix theory, involving ergodic theory and dynamical systems. However, in 2012, Laffey [?] offered a constructive approach to the Boyle-Handelman theorem. He provided a realizing matrix for σ_N and a bound on the number of zeros N required to realize the spectrum by utilizing the matrix

$$X_n = \begin{bmatrix} x_1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ x_2 & x_1 & 2 & 0 & \cdots & \cdots & 0 \\ x_3 & x_2 & x_1 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ x_{n-1} & x_{n-2} & \cdots & \cdots & x_2 & x_1 & n-1 \\ x_n & x_{n-1} & \cdots & \cdots & x_3 & x_2 & x_1 \end{bmatrix},$$

which was previously used to relate the Newton identities, the coefficients of a polynomial and the power sums of its roots.

4 Variations on the NIEP

4.1 SNIEP

The symmetric non-negative inverse eigenvalue problem (SNIEP) adds another restriction to the NIEP. It requires that the realizing matrix of the spectrum σ is symmetric as well as non-negative. Recall that a matrix S is symmetric if $S = S^T$.

When $n = 2$, a solution to the SNIEP can be easily obtained from the NIEP:

Let A $= \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, where $a, b, d \geq 0$.

Thus from equation (3) in section 2.2, we require

$$-b^2 = (a - \lambda_1)(a - \lambda_2)$$

$\implies b = (\lambda_1 - a) = (a - \lambda_2)$
$\implies a = \frac{\lambda_1 + \lambda_2}{2}$

Hence by subbing into the realizing matrix M in section 2.2,

$$S = \begin{bmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2} \\ \frac{\lambda_1-\lambda_2}{2} & \frac{\lambda_1+\lambda_2}{2} \end{bmatrix}$$

is a symmetric matrix that realizes $\sigma = (\lambda_1, \lambda_2)$, where $\lambda_1 \geq |\lambda_2|$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

In his 1974 paper on the SNIEP, Fiedler [?] proved that for every Suleimanova spectrum, there exists a symmetric realizing matrix.

Theorem 4.1 (Šmigoc [?] (2004))**.** *Let A be a non-negative matrix with spectrum $(\lambda_1, \ldots, \lambda_n)$ and diagonal elements $(a_1, \ldots, a_{n-1}, c)$. Let B be a non-negative matrix with Perron eigenvalue c, spectrum $(c, \mu_2, \ldots, \mu_m)$ and diagonal elements $(b_1, \ldots, b_m)$. Then there exists a non-negative matrix C with spectrum $(\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m)$ and diagonal elements $(a_1, \ldots, a_{n-1}, b_1, \ldots, b_m)$. Furthermore, if A and B are symmetric, then C may be chosen to be symmetric also.*

Definition 4.1. The spectrum σ is said to be an element of $\mathcal{H}_n$ if it is possible to construct a non-negative symmetric matrix with spectrum σ by repeated use of the previous theorem, starting with 2×2 matrices as building blocks.

Example 4.1. The matrix

$$A = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix}$$

is a 2×2 non-negative symmetric matrix having spectrum $(8, -2)$.

The matrix

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

is a 2×2 non-negative symmetric matrix having spectrum $(3, -1)$.

Applying the previous theorem reveals the existence of a 3×3 matrix C with spectrum $(8, -1, -2)$ and diagonal elements $(3, 1, 1)$. Hence $C \in \mathcal{H}_n$.

Applying the theorem to C and a 2×2 non-negative symmetric matrix having spectrum $(1, \lambda)$ yields a 4×4 non-negative symmetric matrix $\in \mathcal{H}_n$ having spectrum $(8, \lambda, -1, -2)$. Repeated application of the theorem results in larger non-negative symmetric matrices.

4.2 RNIEP

Another variation of the NIEP is found in the real non-negative inverse eigenvalue problem (RNIEP). In this instance, the list of n eigenvalues σ is required to consist of real numbers. We again look for a $n \times n$ non-negative matrix that realizes σ.

The RNIEP is equivalent to the NIEP when $n = 1$ and $n = 2$, as the solutions from section 2.2 apply when we ask that the eigenvalues be real. Loewy and London (1978) solved the RNIEP for $n \in \{3, 4\}$. The problem for $n \geq 5$ remains unsolved.

It was shown by Johnson, Laffey and Loewy [?] in 1996 that the RNIEP and the SNIEP are equivalent for $n \leq 4$ but that they are not for $n \geq 5$. Hence for any realizable list of real eigenvalues $\sigma = (\lambda_1, \ldots, \lambda_n)$, where $n \leq 4$, there exists also a non-negative symmetric matrix with spectrum σ. Consequently, the SNIEP is also unsolved for $n \geq 5$, though a solution for $n = 5$ when the trace of the realizing matrix is zero was given by Spector [?] in 2011.

5 Ongoing Research

Beyond the fundamental question of solutions to the non-negative inverse eigenvalue problem and its variations, there is additional research on related problems currently in progress. For example, Costa et al. [?] extended the Perron-Frobenius theorem, a central result in matrix theory, to the field $\mathbb{Q}_p$ of p-adic numbers. This extension offers new avenues of analysis for the NIEP outside the usual field of complex numbers.

5.1 Monov's question

Monov [?] asked the following: Given an $n \times n$ non-negative matrix A with characteristic polynomial $f(\lambda)$, is there a non-negative matrix with characteristic polynomial $\frac{\lambda^m}{n} f'(\lambda)$ for some integer $m \geq 0$? In his thesis, Cronin [?] proved that the question can be answered affirmatively for $n \leq 4$ when $m = 0$, and for $n \in \{5, 6\}$ when $m = 0$ and $\text{trace}(A) = 0$.

We now consider the case when $n = 5$ and $\text{trace}(A) > 0$.
Let

$$f(x) = x^5 + p_1 x^4 + p_2 x^3 + p_3 x^2 + p_4 x + p_5,$$

be the characteristic polynomial of a 5×5 non-negative matrix, and let

$$g(x) = \frac{f'(x)}{5} = x^4 + q_1 x^3 + q_2 x^2 + q_3 x + q_4,$$

where $q_1 = \frac{4p_1}{5}, q_2 = \frac{3p_2}{5}, q_3 = \frac{2p_3}{5}, q_4 = \frac{p_4}{5}$.

Newton's identities are a useful tool in this branch of matrix theory. In terms of the NIEP, we can use the identities to relate the coefficients of a characteristic polynomial to the power sums s_k. Given a characteristic polynomial $p(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$, they tell us that

$$-a_1 = s_1$$

$$2a_2 = -a_1 s_1 - s_2$$

$$-3a_3 = a_2 s_1 + a_1 s_2 + s_3$$

$$4a_4 = -a_3 s_1 - a_2 s_2 - a_1 s_3 - s_4$$

Applying Newton's identities to $f(x)$:

$$p_1 = -s_1$$
$$p_2 = \frac{s_1^2 - s_2}{2}$$
$$p_3 = \frac{-s_1^3}{6} + \frac{s_1 s_2}{2} - \frac{s_3}{3}$$
$$p_4 = \frac{s_1^4}{24} + \frac{s_2^2 - 2s_1^2 s_2}{8} + \frac{s_1 s_3}{3} - \frac{s_4}{4}$$

where $s_k = \sum_{i=1}^5 \lambda_i^k$, and $\lambda_1, \dots, \lambda_5$ are the roots of $f(x)$.
Applying the identities to $g(x)$:

$$q_1 = \frac{4p_1}{5} = \frac{-4s_1}{5} = -S_1 \implies S_1 = \frac{4s_1}{5}$$

$$q_2 = \frac{3p_2}{5} = \frac{3s_1^2 - 3s_2}{10} = \frac{S_1^2 - S_2}{2} = \frac{\frac{16s_1^2}{25} - S_2}{2} \implies S_2 = \frac{s_1^2 + 15s_2}{25}$$
$$q_3 = \frac{2p_3}{5} = \frac{-s_1^3 + 3s_1 s_2 - 2s_3}{15} = \frac{-S_1^3}{6} + \frac{S_1 S_2}{2} - \frac{S_3}{3} \implies S_3 = \frac{-s_1^3 + 15s_1 s_2 + 50s_3}{125}$$
$$q_4 = \frac{p_4}{5} = \frac{s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 - 3s_2^2 - 6s_4}{120} = \frac{S_1^4}{24} + \frac{S_2^2 - 2S_1^2 S_2}{8} + \frac{S_1 S_3}{3} - \frac{S_4}{4}$$
$$\implies S_4 = \frac{2447s_1^4 - 300s_1^2 s_2 + 1500s_1 s_3 + 750s_2^2 + 1875s_4}{9375}$$

where $S_k = \sum_{i=1}^4 \mu_i^k$, and $\mu_1, \dots, \mu_4$ are the roots of $g(x)$.

To determine realizability when $n = 4$, we must consider the inequality

$$3S_2 \geq S_1^2$$
$$\iff 45s_2 \geq 13s_1^2$$

We have the following:

$$\text{trace}(A) = s_1 > 0$$

From [?]:

$$25s_3 - 15s_1 s_2 + 2s_1^3 + \frac{3}{2}(5s_2 - s_1^2)^{\frac{3}{2}} \geq 0$$

By the JLL inequalities:

$$5s_2 \geq s_1^2$$
$$25s_3 \geq s_1^3$$
$$5s_4 \geq s_2^2$$
$$125s_4 \geq s_1^4$$

It remains to show that

$$15s_1 s_2 + 50s_3 \geq s_1^3$$

and

$$9758s_1^4 + 6000s_1 s_3 + 7500s_4 \geq 2100s_1^2 s_2 + 3750s_2^2.$$

This problem will require further work and inequalities linking $s_k s_j$ and s_k for $k, m = 1, 2, 3, 4$ would be desirable. Monov [?] also put forward the following conjecture.

Conjecture 5.1 (Monov). *Let A be a real non-negative matrix and let $f(\lambda) = \det(\lambda I - A)$ be the characteristic polynomial of A. Then*

$$s'_k = \sum_{i=1}^{n-1} \mu_i^k \geq 0 \ \forall\, k \in (1, 2, \ldots)$$

where $\mu_1, \ldots, \mu_{n-1}$ are the roots of $f'(\lambda)$.

A counterexample was provided by Cronin [?] to disprove the related question of whether the property of a polynomial's roots having positive power sums s_k is inherited by the polynomial's derivative.

The following conjecture is credited to Johnson in [?].

Conjecture 5.2 (Johnson). *Let $\sigma = (\lambda_1, \ldots, \lambda_n)$, with $n \geq 2$, be realizable and let $p(x) = \prod_{i=1}^{n}(x - \lambda_i)$. Then $\sigma' = (\mu_1, \ldots, \mu_{n-1})$ is realizable, where $\mu_1, \ldots, \mu_{n-1}$ are the roots of $p'(x)$.*

This conjecture implies Conjecture 5.1, as a necessary condition for realizability is that $s_k \geq 0$ (condition I).

In a recent paper, Hoover et al. [?] answered Johnson's conjecture affirmatively, and thus Monov's, in several particular cases. For example, if $C(p) \geq 0$, then $C(p)$ realizes σ, and σ' is realizable by a companion matrix also.

Proof. Let

$$p(x) = x^n + p_1 x^{n-1} + \cdots + p_{n-1}x + p_n,$$

where $n \in \mathbb{N}$.
If

$$C(p) \geq 0$$

then

$$p_i \leq 0 \ \forall\, i \in (1, \ldots, n).$$

Then

$$p'(x) = nx^{n-1} + (n-1)p_1 x^{n-2} + \cdots + 2p_{n-2}x + p_{n-1}.$$

It is clear that $p'(x)$ and $\frac{1}{n}p'(x)$ have the same list of roots σ'. $C(\frac{1}{n}p') \geq 0$ since the coefficients of $\frac{1}{n}p'(x)$ are non-positive, which follows from the fact that $p_i \leq 0 \ \forall\, i \in (1, \ldots, n)$. □

The conjectures are also shown to hold when σ is a Suleimanova spectrum. In fact, σ' is also a Suleimanova spectrum. Both conjectures remain open and are an example of ongoing research on the non-negative inverse eigenvalue problem.