Inequalities

Marius Ghergu

School of Mathematics and Statistics

University College Dublin

The Cauchy-Schwarz Inequality:

For any real numbers

$$a_1, a_2, \ldots, a_n \quad \text{and} \quad b_1, b_2, \ldots, b_n$$

we have

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1 b_1 + a_2 b_2 + \ldots a_n b_n)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Proof. Consider the quantity

$$F(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2 \geq 0 \quad \text{for all } x \in \mathbb{R}.$$

Expanding the brackets we have

$$F(x) = (a_1^2 + a_2^2 + \cdots + a_n^2)x^2 - 2(a_1b_2 + a_2b_2 + \ldots a_nb_n)x + (b_1^2 + b_2^2 + \cdots + b_n^2),$$

that is,

$$F(x) = Ax^2 - 2Bx + C \geq 0 \quad \text{for all } x \in \mathbb{R},$$

where

$$A = a_1^2 + a_2^2 + \cdots + a_n^2,$$

$$B = a_1b_2 + a_2b_2 + \ldots a_nb_n,$$

$$C = b_1^2 + b_2^2 + \cdots + b_n^2.$$

This implies that $(2B)^2 - 4AC \leq 0$ which yields $AC \geq B^2$. Hence

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_2 + a_2b_2 + \ldots a_nb_n)^2.$$

The equality holds when there exists $x \in \mathbb{R}$ such that $F(x) = 0$ so

$$a_1x - b_1 = a_2x - b_2 = \cdots = a_nx - b_n = 0,$$

which implies $x = \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \cdots = \dfrac{a_n}{b_n}$.

Problem 1. Prove that for any real numbers $a_1, a_2, \dots, a_n$ we have

$$n(a_1^2 + a_2^2 + \cdots + a_n^2) \geq (a_1 + a_2 + \dots a_n)^2.$$

Solution: Apply Cauchy-Schwarz inequality with $b_1 = b_2 = \cdots = b_n = 1$.

Problem 2. (Stanford Maths Tournament 2022)
Let x, y, z be real numbers such that $x^2 + 2y^2 + 3z^2 = 96$. Find the maximum and the minimum of $x + 2y + 3z$.

Solution: By the Cauchy-Schwarz inequality:

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

with equality if and only if $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$. We ask ourselves how to apply the above inequality so as to get

$$(\qquad\qquad)(x^2 + 2y^2 + 3z^2) \geq (x + 2y + 3z)^2.$$

We identify

$$(1 + 2 + 3)(x^2 + 2y^2 + 3z^2) \geq (x + 2y + 3z)^2.$$

Hence

$$6 \times 96 \geq (x + 2y + 3z)^2 \implies (x + 2y + 3z)^2 \leq 24^2$$

Hence

$$|x + 2y + 3z)| \leq 24 \implies -24 \leq x + 2y + 3z \leq 24.$$

The equality holds if

$$\frac{x^2}{1} = \frac{2y^2}{2} = \frac{3z^2}{3} \implies x^2 = y^2 = z^2 \implies |x| = |y| = |z| = 4.$$

The maximum of $x + 2y + 3z$ is 24 and occurs for $x = y = z = 4$.
The minimum of $x+2y+3z$ is -24 and occurs for $x = y = z = -4$.

Problem 3. (Dublin Area Selection Test 2015)

Let $x, y, z, w > 0$ and suppose that $xyzw = 16$. Show that

$$\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x} \geq 4$$

with equality only when $x = y = z = w = 2$.

Solution: The Cauchy inequality gives

$$\left(\left(\frac{x}{\sqrt{x+y}}\right)^2 + \left(\frac{y}{\sqrt{y+z}}\right)^2 + \left(\frac{z}{\sqrt{z+w}}\right)^2 + \left(\frac{w}{\sqrt{w+x}}\right)^2\right) \times$$

$$\left((\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+w})^2 + (\sqrt{w+x})^2\right)$$

$$\geq (x+y+z+w)^2,$$

with equality only when $x = y = z = w$. This simplifies to:

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \cdot 2(x+y+z+w) \geq (x+y+z+w)^2$$

and hence

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq \frac{x+y+z+w}{2}.$$

Applying the AM-GM to the right-hand term gives

$$\left(\frac{x^2}{x+y} + \frac{y^2}{y+z} + \frac{z^2}{z+w} + \frac{w^2}{w+x}\right) \geq 2\sqrt[4]{xyzw}$$

with equality only when $x = y = z = w$. Since $xyzw = 16$, the result follows at once.

Problem 3. Prove that for any real numbers $a_1, a_2, \ldots, a_n$ and for any positive numbers $x_1, x_2, \ldots, x_n$ we have

$$\frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \cdots + \frac{a_n^2}{x_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{x_1 + x_2 + \cdots + x_n}.$$

Solution: Apply Cauchy-Schwarz inequality for $a_1, a_2, \ldots, a_n$ and $b_1 = \sqrt{x_1}, b_2 = \sqrt{x_2}, \ldots, b_n = \sqrt{x_n}$.

Note. We could also use Induction Principle over the number $n \geq 2$ to prove this result.

Problem 4. (IMO 1995)

Let a, b, c be three postive numbers such that $abc = 1$.

Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution: Denote $x = \frac{1}{a}$, $y = \frac{1}{b}$ and $z = \frac{1}{c}$.

Since $abc = 1$ we have $xyz = 1$ and our inequality to prove becomes

$$\frac{1}{\frac{1}{x^3}\left(\frac{1}{y} + \frac{1}{z}\right)} + \frac{1}{\frac{1}{y^3}\left(\frac{1}{z} + \frac{1}{x}\right)} + \frac{1}{\frac{1}{z^3}\left(\frac{1}{x} + \frac{1}{y}\right)} \geq \frac{3}{2},$$

That is (because $xyz = 1$)

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}. \qquad (1)$$

Apply Cauchy-Schwarz inequality for

$$a_1 = \frac{x}{\sqrt{y+z}}, \quad a_2 = \frac{y}{\sqrt{z+x}}, \quad a_3 = \frac{z}{\sqrt{x+y}}$$

$$b_1 = \sqrt{y+z}, \quad b_2 = \sqrt{z+x}, \quad z = \sqrt{x+y}.$$

Thus,

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \geq (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

becomes

$$\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) \cdot 2(x+y+z) \geq 3(x+y+z).$$

which simplifies to (1).

Problem 5. (Iran Math Olympiad 1998)

Let $x, y, z > 1$ be such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$.

Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Solution: Note that $\dfrac{x-1}{x} + \dfrac{y-1}{y} + \dfrac{z-1}{z} = 1.$

Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{\frac{x-1}{x}}, \quad a_2 = \sqrt{\frac{y-1}{y}}, \quad a_3 = \sqrt{\frac{z-1}{z}}$$

$$b_1 = \sqrt{x}, \quad b_2 = \sqrt{y}, \quad b_3 = \sqrt{z}.$$

Hence

$$(x+y+z)\Big(\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}\Big) \geq \Big(\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\Big)^2,$$

so

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Problem 6.

Let $a, b, c > 0$ be such that

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \geq 1.$$

Prove that

$$a + b + c \geq ab + bc + ca.$$

Solution: Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{a}, \quad a_2 = \sqrt{b}, \quad a_3 = 1$$

$$b_1 = \sqrt{a}, \quad b_2 = \sqrt{c}, \quad b_3 = c.$$

We find

$$(a+b+1)(a+b+c^2) \geq (a+b+c)^2$$

that is,

$$\frac{1}{a+b+1} \leq \frac{a+b+c^2}{(a+b+c)^2}.$$

Similarly,

$$\frac{1}{b+c+1} \leq \frac{a^2+b+c}{(a+b+c)^2} \quad \text{and} \quad \frac{1}{c+a+1} \leq \frac{a+b^2+c}{(a+b+c)^2}.$$

Adding up these last three inequalities and using our hypothesis we find

$$1 \leq \frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq \frac{a^2+b^2+c^2+2(a+b+c)}{(a+b+c)^2},$$

so

$$a^2+b^2+c^2+2(a+b+c) \geq (a+b+c)^2,$$

which yields $a + b + c \geq ab + bc + ca$.

Problem 7. (German Math Olympiad)

Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ be positive numbers with sum S. Prove that

$$\frac{x_1}{S-x_1} + \frac{x_2}{S-x_2} + \cdots + \frac{x_n}{S-x_n} \geq \frac{n}{n-1}.$$

Solution: Apply Cauchy-Schwarz inequality for

$$a_1 = \sqrt{\frac{x_1}{S-x_1}}, \quad a_2 = \sqrt{\frac{x_2}{S-x_2}}, \quad \ldots, \quad a_n = \sqrt{\frac{x_n}{S-x_n}}$$

$$b_1 = \sqrt{x_1(S-x_1)}, \quad b_2 = \sqrt{x_2(S-x_2)}, \quad \ldots, \quad b_n = \sqrt{x_n(S-x_n)}.$$

Note that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = \frac{x_1}{S-x_1} + \frac{x_2}{S-x_2} + \cdots + \frac{x_n}{S-x_n}$$

$$\begin{aligned} b_1^2 + b_2^2 + \cdots + b_n^2 &= x_1(S-x_1) + x_2(S-x_2) + \cdots + x_n(S-x_n) \\ &= S(x_1 + x_2 + \cdots + x_n) - (x_1^2 + x_2^2 + \cdots + x_n^2) \\ &= S^2 - T, \end{aligned}$$

where $T = x_1^2 + x_2^2 + \cdots + x_n^2$. Also, $a_1 b_1 + a_2 b_2 + \ldots a_n b_n = S$

Thus, by Cauchy-Schwarz inequality we find

$$(S^2 - T)\left(\frac{x_1}{S-x_1} + \frac{x_2}{S-x_2} + \cdots + \frac{x_n}{S-x_n}\right) \geq S^2$$

Hence

$$\frac{x_1}{S-x_1} + \frac{x_2}{S-x_2} + \cdots + \frac{x_n}{S-x_n} \geq \frac{S^2}{S^2 - T}.$$

It remains to prove that $\dfrac{S^2}{S^2 - T} \geq \dfrac{n}{n-1}$ or even

$$nT \geq S^2 \Longleftrightarrow n(x_1^2 + x_2^2 + \cdots + x_n^2) \geq (x_1 + x_2 + \cdots + x_n)^2.$$

This last inequality follows again from the Cauchy-Schwarz ineq applied to

$$a_1 = x_1, \quad a_2 = x_2, \quad \ldots, \quad a_n = x_n$$

$$b_1 = b_2 = \cdots = b_n = 1.$$