WHEN BOTH SIDES AGREE

• Quadratic

$$(x-a)(x-b) = x^2 - (a+b)x + ab$$

Special cases:

$$b = a: \quad (x-a)^2 = x^2 - 2ax + a^2$$
$$b = -a \quad (x-a)(x+a) = x^2 - a^2$$

• Qubic

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$
$$= (a-b)^3 + 3ab(a-b)$$

$$a^3 + b^3 = (a+b)(a^2-ab+b^2)$$
$$= (a+b)^3 - 3ab(a+b)$$

PROOF : Expand & compare.

$$(a-b)\underbrace{(a^2+ab+b^2)}_{\text{complete to } (a-b)^2}$$

$$= (a-b)(a^2-2ab+b^2+3ab)$$
$$= (a-b)((a-b)^2+3ab)$$

*

$$a^3+b^3+c^3-3abc = (a+b+c)\cdot(a^2+b^2+c^2-ab-bc-ac)$$

PF : $a^3+b^3+c^3-3abc$

$$= \underbrace{a^3+3a^2b+3ba^2+b^3}+c^3-3ab(a+b)-3abc$$
* $$= \underbrace{(a+b)^3+c^3}-3ab(a+b+c)$$
$$= \underbrace{(a+b+c)^3-3(a+b)\cdot c(a+b+c)}-3ab(a+b+c)$$
$$= (a+b+c)\left((a+b+c)^2-3(a+b)\cdot c-3ab\right)$$

Quartic

$$x^4 + 4y^4 = (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$$

Sophie - Germain Identity

PF: $x^4 + 4y^4 = x^4 + 4x^2y^2 - 4x^2y^2 + 4y^4$

$$= (x^2 + 2y^2)^2 - 4x^2y^2$$

$$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$$

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

$$= (ac - bd)^2 + (ad + bc)^2$$

Brahmagupta-Fibonacci Identity

Degree n

$$a^n - b^n = (a - b) \cdot (a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1}), \quad n \in \mathbb{N}$$

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + \ldots - ab^{n-2} + b^{n-1}), \quad \underline{n\text{-odd}}$$

Q) Evaluate $(2+1)(2^2+1)(2^4+1)\dots(2^{2^{10}}+1)+1$

Sol. Note: $1=(2-1)$

$$(2-1)(2+1)(2^2+1)(2^4+1)\dots(2^{2^{10}}+1)+1$$

$$=\underbrace{(2^2-1)(2^2+1)}_{2^4-1}(2^4+1)\dots(2^{2^{10}}+1)+1$$

$$\dots \quad = (2^{2^{10}}-1)(2^{2^{10}}+1)+1$$

$$= 2^{2^{11}}-1+1 = 2^{2^{4}}$$

Q) Compute, $n \in \mathbb{N}$

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\dots+\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}=S$$

Sol. $$\frac{1}{\sqrt{k}+\sqrt{k+1}}=\frac{\sqrt{k+1}-\sqrt{k}}{1}$$

$$S=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\dots+\underbrace{\sqrt{n^2}}_{n}-\sqrt{n^2-1}$$

$$= n-1$$

Q) Given $x+\frac{1}{x}=3$. Find the value of

(i) $x^3+\frac{1}{x^3}$

(ii) $x^4+\frac{1}{x^4}$

Sol.

(i) $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$=\left(x+\frac{1}{x}\right)\left(\left(x+\frac{1}{x}\right)^2-3\right)$

$=3\cdot(9-3)=3\cdot 6=18$

(ii) $x^4+\frac{1}{x^4}=\left(x^2+\frac{1}{x^2}\right)^2-2$

$=\left(\left(x+\frac{1}{x}\right)^2-2\right)^2-2=(9-2)^2-2$

$=47$

Completing the squares can help us determine extreme values.

Q) Find the smallest possible value that

$$3a^2+27b^2+5c^2-18ab-30c+237$$

can take for real a, b, c.

Sol. $3a^2+27b^2+5c^2-18ab-30c+237$

$= 3a^2+3\cdot 9b^2-3\cdot 6ab+5c^2-5\cdot 6c+237$

$= 3(a^2-6ab+9b^2)+5c^2-5\cdot 6c+5\cdot 9$

$+237-45$

$= 3(a-3b)^2+5(c-3)^2+192 \geq 192$

Equality for $a=3b$ and $c=3$

Q) Given that $a, b, c \in \mathbb{R}$ satisfy $a^3 + b^3 + 3ab = 1$, find $a+b$

Sol. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$

$a^3 + b^3 = (a+b)((a+b)^2 - 3ab)$

From $a^3 + b^3 + 3ab - 1 = 0$

we get

$(a+b)((a+b)^2 - 3ab) + 3ab - 1 = 0$

set $(a+b) = x$

$x^3 - 3ab\,x + 3ab - 1 = 0$

$x^3 - 1 - 3ab(x-1) = 0$

$(x-1)\cdot(x^2 + x + 1 - 3ab) = 0$

<u>One solution</u>: $x = 1$

$x^2 + x + 1 - 3ab = 0$

$(a+b)^2 + (a+b) + 1 - 3ab = 0$

$a^2 + 2ab + b^2 + a + b + 1 - 3ab = 0$

$a^2 - ab + b^2 + a + b + 1 = 0 \;/\cdot 2$

$2a^2 - 2ab + 2b^2 + 2a + 2b + 2 = 0$

$(a-b)^2 + (a+1)^2 + (b+1)^2 = 0$

$\Rightarrow a = b \quad a = -1 \quad b = -1$

<u>Another solution</u> $a + b = -2$

Q) Find all nonnegative integers n : so that $1+4\cdot 9^{2n}$ is prime.

Sol. $a = 1 + 4\,3^{4n} = 1^4 + 4\cdot(3^n)^4$

$= (1 + 2\cdot 3^{2n} + 2\cdot 3^n)(1 + 2\cdot 3^{2n} - 2\cdot 3^n)$

a - prime $\Rightarrow 1+2\cdot 3^{2n} - 2\cdot 3^n = 1$

$\Rightarrow 2\cdot 3^{2n} - 2\cdot 3^n = 0$

$\Rightarrow n = 0$