THE PIGEONHOLE PRINCIPLE

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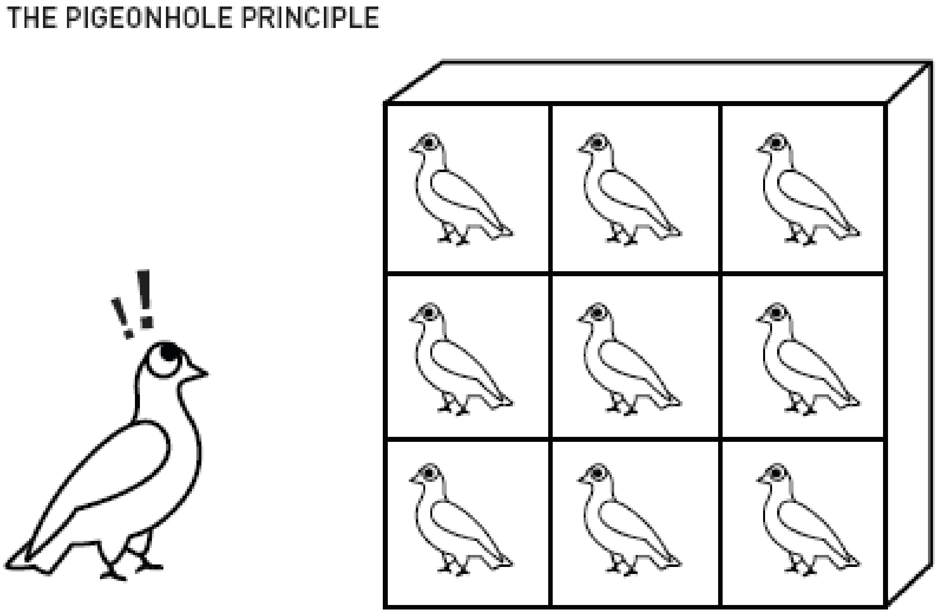
The Pigeonhole Principle: If $n+1$ objects are placed into n boxes, then some box contains at least 2 objects.

Proof: Suppose that each box contains at most one object. Then there must be at most n objects in all. But this is false, since there are $n+1$ objects. Thus some box must contain at least 2 objects.

This combinatorial principle was first used explicitly by Dirichlet (1805-1859). Even though it is extremely simple, it can be used in many situations, and often in *unexpected* situations. Note that the principle asserts the *existence* of a box with more than one object, but does not tell us anything about which box this might be.

In problem solving, the difficulty of applying the pigeonhole principle consists in figuring out which are the 'objects' and which are the 'boxes'.



Problem 1. Prove that among 13 people, there are two born in the same month.

Solution. There are $n = 12$ months ('boxes'), but we have $n+1 = 13$ people ('objects'). Therefore two people were born in the same month.

Problem 2. Prove that if we are given 5 points in the plane with integer coordinates, we can choose two of them so that the midpoint also has integer coordinates.

Solution. Note that the midpoint of (x, y) and (z, w) is $\left(\frac{x+z}{2}, \frac{y+w}{2}\right)$.

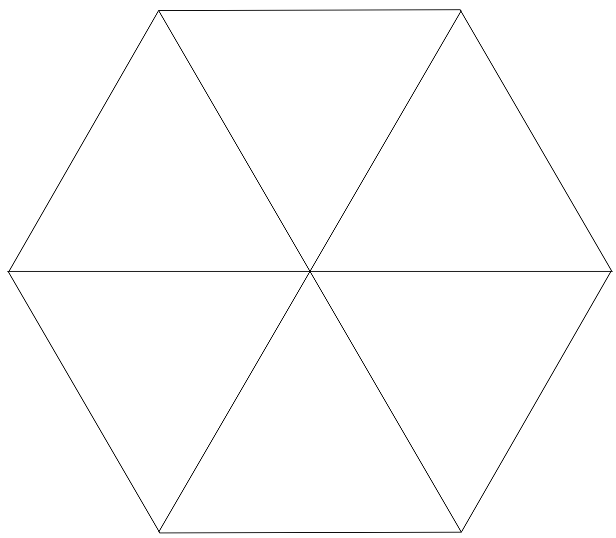
We can label the coordinates of the points as being either (even, even), (even, odd), (odd, even), or (odd, odd). By the pigeonhole principle, two of the points (say $P = (x, y)$ and $Q = (z, w)$) have the same label.

But then $x + z$ and $y + w$ are both even, so the midpoint of PQ has integer coordinates.

Exercise 1. How many people do you need to be able to say with certainty that two have the same birthday?

Problem 3. Seven points lie inside a hexagon of side length 1. Show that two of the points whose distance apart is at most 1.

Solution. Partition the hexagon into six parts as shown below. Now there are six parts (boxes), into which seven points (objects) are distributed. So some part contains at least 2 points. These points must be within distance 1 of each other.



Exercise 2. Six points lie inside a rectangle of dimensions 3×4. Show that two of the points are at most a distance $\sqrt{5}$ apart.

Problem 4. Show that given a set S of 10 positive integers each having two digits, there are two disjoint nonempty subsets A and B of S that have the same sum of elements. [Note: "disjoint" means having no elements in common.]

Solution. The two-digit numbers must lie between 10 and 99 inclusive. The possible sums range from 10 to $90 + 91 + \cdots + 99 = 10 \cdot 90 + (1 + 2 + \cdots + 9) = 900 + (9 \cdot 10)/2 = 945$. So there are 936 possible sums.

On the other hand, there are $2^{10} - 1 = 1023$ nonempty subsets of the 10 numbers.

Since there are 1023 nonempty subsets and 936 possible sums, by the pigeonhole principle, there must be two distinct subsets A and B with the same sum.

If A and B are disjoint, then we are finished. If not, we can simply remove the common elements from sets A and B and we produce two disjoint subsets $A' = A - (A \cap B)$ and $B' = B - (A \cap B)$ having the same sum.

Note that the new subsets A' and B' are also nonempty, because if $A = A \cap B$ or $B = A \cap B$, then one of A and B is a subset of the other, but this is not possible since they have the same sum.

Problem 5. Let n be a positive integer that is not divisible by 2 or 5. Prove that there is a multiple of n whose decimal representation consists entirely of ones.

Solution. Make a list of n numbers 1, 11, 111, 1111, $\ldots$, $11111\ldots111$, where the final number in the list has n ones. Now, consider the remainders when each of these numbers is divided by n.

If one of these remainders is 0, then we are finished.

If this does not happen, then we will have n numbers but only $n-1$ possible remainders. It follows by the pigeonhole principle that two of the numbers in our list give the same remainder when divided by n.

But this means that their difference, which is a number of the form $m = 1111\ldots10000\ldots0$ is divisible by n. We can get rid of the trailing zeros by dividing m by 10^k, where k is the number of trailing zeros, to get a number q which consists entirely of ones. We will now show that q is the number we are looking for.

Since n divides into $m = 10^k q = 2^k 5^k q$, but n is not divisible by 2 or 5, n must divide into q.

Problem 6. Suppose we have 27 *distinct* odd positive integers all less than 100. ['Distinct' means that no two numbers are equal]. Show that there is a pair of numbers whose sum is 102. What if there were only 26 odd positive integers?

Solution. There are 50 positive odd numbers less than 100:

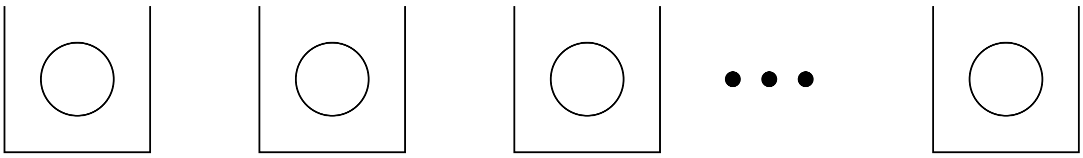
$$\{1, 3, 5, \cdots, 99\} .$$

We can partition these into subsets as follows:

$$\{1\}, \{3, 99\}, \{5, 97\}, \{7, 95\}, \{9, 93\}, \cdots, \{49, 53\}, \{51\}.$$

Note that the sets of size 2 have elements which add to 102. There are 26 subsets (boxes) and 27 odd numbers (objects). So at least two numbers (in fact, exactly two numbers) must lie in the same subset, and therefore these add to 102.

Note on the pigeonhole principle: What if n objects are placed in n boxes? Well, then we *cannot* assert that some box contains at least 2 objects. But note that the only way this can be avoided is if *all* of the boxes contain *exactly one* object.



The "Generalized" Pigeonhole Principle: If $kn + 1$ objects are placed in n boxes, then some box contains at least $k+1$ objects.

Proof: Suppose that each box contains at most k objects. Then there must be at most kn objects in all. But this is false, since there are $kn + 1$ objects. Thus some box must contain at least $k + 1$ objects.

Problem 7. Show that in a group of 15 people, at least three were born on the same day of the week.

Solution. We have $15 = 2(7)+1$ people (objects), and 7 weekdays (boxes). Here $k = 2$, $n = 7$. Therefore three people were born in the same day of the week.

Problem 8. 41 rooks are placed on a 10×10 chessboard. Prove that you can choose 5 of them that do not attack each other. [Note: we say that two rooks "attack each other" if they are both in the same row or column of the chessboard.]

"Clunky but Nice" Solution: First, there are 10 rows of the chessboard, and we are placing 41 rooks. Since $41 = 4 \cdot 10 + 1$, by the pigeonhole principle there must be one row among these 10 that contains at least 5 rooks. Call this row R_1.

Now, let's forget about row R_1 for the moment and consider the other 9 rows. On the 9 rows, we must place at least 31 rooks. Since $31 = 3 \cdot 9 + 4$, by the pigeonhole principle there must be one row among these 9 that contains at least 4 rooks. Call this row R_2.

Now, let's forget about rows R_1 and R_2 and consider the other 8 rows. On the 8 rows, we must place at least 21 rooks. Since $21 = 2 \cdot 8 + 5$, by the pigeonhole principle there must be one row among these 8 that contains at least 3 rooks. Call this row R_3.

Now, let's forget about rows R_1, R_2 and R_3, and consider the other 7 rows. On the 7 rows, we must place at least 11 rooks. By the pigeonhole principle there must be one row among these 7 that contains at least 2 rooks. Call this row R_4.

Finally, we observe that even if rows R_1, R_2, R_3 and R_4 are completely filled, we still have one more rook to place. So there must be one row among the remaining 6 that contains at least 1 rook. Call this row R_5.

Now, we work backwards to identify the 5 non-attacking rooks that we want. Start by picking the rook in row R_5 (call this X_1).

Next, consider row R_4. There are at least 2 rooks in this row, and both of them cannot attack rook X_1. So we choose the one that is not attacking X_1 (let's call this rook X_2). If neither rook is attacking X_1, we have a free choice.

Next, consider row R_3. There are at least 3 rooks in this row, and so at least one of these three must be such that it does not attack rooks X_1 or X_2. So we choose this rook and call it X_3.

Next, consider row R_2. There are at least 4 rooks in this row, and so at least one of these four must be such that it does not attack rooks X_1, X_2 or X_3. So we choose this rook and call it X_4.

Finally, consider row R_1. There are at least 5 rooks in this row, and so at least one of these five must be such that it does not attack rooks X_1, X_2, X_3 or X_4. So we choose this rook and call it X_5.

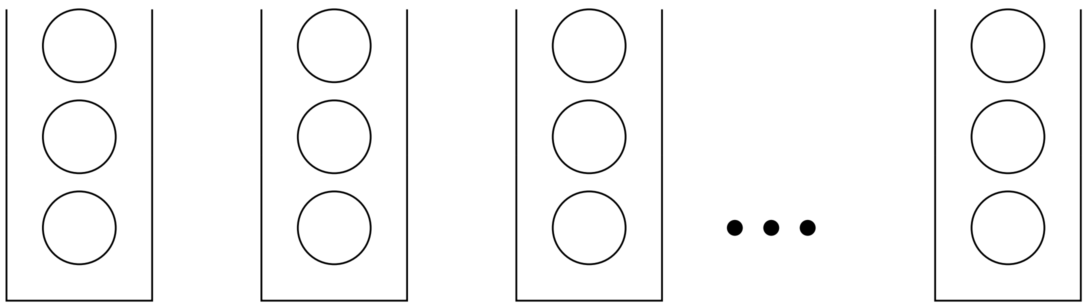
By using this procedure, we have identified the required 5 non-attacking rooks (X_1, X_2, X_3, X_4, X_5) and the result is proved.

"Slick" Solution: Label the 100 squares of the chessboard with the numbers from 1 to 10 as follows:

1	2	3	4	5	6	7	8	9	10
10	1	2	3	4	5	6	7	8	9
9	10	1	2	3	4	5	6	7	8
8	9	10	1	2	3	4	5	6	7
7	8	9	10	1	2	3	4	5	6
6	7	8	9	10	1	2	3	4	5
5	6	7	8	9	10	1	2	3	4
4	5	6	7	8	9	10	1	2	3
3	4	5	6	7	8	9	10	1	2
2	3	4	5	6	7	8	9	10	1

Each rook can also be considered to have a label, which is simply the label of the square on which it is placed. Now we have 41 rooks but only 10 different labels. So, by the pigeonhole principle (since $41 = 4 \cdot 10 + 1$) there must be 5 rooks all of which have the same label. But it is easy to see that these 5 rooks do not attack each other, as they lie on a "diagonal" of the board.

Note on the generalized pigeonhole principle: What if kn objects are placed in n boxes? This means that we cannot assert that some box contains at least $k+1$ objects. But note that the only way this can be avoided is if *all* of the boxes contain *exactly* k objects.



Exercise 3. How many people need to be present in order to be able to assert with certainty that three have the same birthday?

Exercise 4. Seven boys and five girls are seated (in an equally spaced fashion) around a table with 12 chairs. Prove that there are two boys sitting opposite each other.

Exercise 5. Each square of a 3×7 board is coloured black or white. Prove that, for any such colouring, the board contains a subrectangle whose four corners are the same colour.

Exercise 6. 342 points are selected inside a cube of side length 7. Can you place a small cube with side length 1 inside the big cube such that the interior of the small cube does not contain one of the selected points?

Exercise 7. Prove that however one selects 55 distinct integers $1 \leq x_1 < x_2 < x_3 < ... < x_{55} \leq 100$, there will be a pair that differ by 9, a pair that differ by 10, a pair that differ by 12, and a pair that differ by 13. Show also that (surprisingly!) there need not be a pair of numbers that differ by 11.

Exercise 8. The *digital sum* of a number is defined as the sum of its decimal digits. For example, the digital sum of 386 is $3+8+6=17$.

(a): 35 two-digit numbers are selected. Prove that there are three of them with the same digital sum.

(b): 168 three-digit numbers are selected. Prove that it is possible to find eight of them of them with the same digital sum.

Note that in the above, the first digit of a number is not allowed to be 0.

Exercise 9. In a meeting, there are representatives of n countries ($n \geq 2$) sitting at a round table. It is observed that for any two representatives of the same country, their neighbours to their right cannot belong to the same country. Find the largest possible number of representatives at the meeting.

Exercise 10. Given a regular 2007-gon, find the smallest positive integer k such that among any k vertices of the polygon there are 4 with the property that the convex quadrilateral they form shares 3 sides with the polygon.