Combinatorics Exercises and Solutions: Saturday October 25th, 2025

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Pigeonhole Principle

(1) Given any 52 integers, show that there is some pair among them whose sum or whose difference is divisible by 100.

Solution: Put the 52 numbers in the 51 pigeonholes

$$\{0\}, \{1, 99\}, \{2, 98\}, \ldots, \{k, 100 - k\}, \ldots, \{49, 51\}, \{50\}$$

by assigning to each number its remainder on division by 100. By the P.P., some pair of numbers are assigned to the same pigeonhole. If they both have the same remainder on division by 100 then their difference is divisible by 100. Otherwise the remainders sum to 100 and hence the sum of the two numbers is divisible by 100.

(**Remark 1:** Of course, this problem generalizes: Among any $n + 2$ numbers, there is some pair whose sum or difference is divisible by $2n$.

Remark 2: By considering the 51 numbers $0, 1, \ldots, 50$, no pair of which has a difference or sum divisble by 100, we see that 52 is smallest number which will guarantee the conclusion in the problem.)

(2) Show that among any 17 integers there is a set of 5 whose sum is divisible by 5.

Solution: We consider remainders on division by 5.

If all 5 possible remainders occur among the 17 integers, then for $i = 0, 1, 2, 3, 4$ there is a number n_i among the 17 with remainder i on division by 5. It follows that $n_0 + n_1 + \cdots + n_4$ leaves the same remainder as $0 + 1 + \cdots + 4 = 10$ on division by 5 and so this sum is divisible by 5.

On the other hand, if there is a remainder which does not occur, then all 17 numbers/pigeons are assigned to the 4 remaining possible remainders/pigeonholes. By the *strong pigeonhole principle* some set of 5 of the 17 numbers have the same remainder (since $4 \cdot 4 = 16 < 17$), and hence the sum of these is divisible by 5.

(**Remark:** This generalizes of course: If n is an odd number then among any $(n-1)^2+1$ integers there is a subset of size n whose sum is divisble by n.)

Binomial Coefficients

(1) Show that $\binom{n}{1} + 2\binom{n}{2} + \cdots + k\binom{n}{k} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$.

Solution 1: Let $S := 0\cdot\binom{n}{0}+1\cdot\binom{n}{1}+2\binom{n}{2}+\cdots+k\binom{n}{k}+\cdots+n\binom{n}{n} = n2^{n-1}$.

Since $\binom{n}{k} = \binom{n}{n-k}$ for $k \leq n$, we have

$$\begin{aligned} S &= 0 \cdot \binom{n}{n} + 1 \cdot \binom{n}{n-1} + \cdots k \cdot \binom{n}{n-k} + \cdots + n \cdot n0 \\ &= n \cdot \binom{n}{0} + \cdots + (n-k) \cdot \binom{n}{k} + \cdots + 0 \cdot \binom{n}{n}. \end{aligned}$$

Thus

$$\begin{aligned} 2S = S + S &= 0 \cdot \binom{n}{0} + \binom{n}{1} + 2\binom{n}{2} + \cdots + k\binom{n}{k} + \cdots + n\binom{n}{n} \\ &+ n \cdot \binom{n}{0} + \cdots + (n-k) \cdot \binom{n}{k} + \cdots + 0 \cdot \binom{n}{n} \\ &= n \cdot \binom{n}{0} + \cdots + (k + (n-k))\binom{n}{k} + \cdots + n \cdot \binom{n}{n} \\ &= n\left(\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n}\right) = n2^n. \end{aligned}$$

Dividing both sides of $2S = n2^n$ by 2 gives the required formula.

Solution 2: The binomial theorem tells us that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n.$$

Differentiating both sides with respect to x then gives:

$$n(1+x)^{n-1} = \binom{n}{1} + \cdots + k\binom{n}{k}x^{k-1} + \cdots + n\binom{n}{n}x^{n-1}.$$

Setting $x = 1$ then gives

$$n2^{n-1} = \binom{n}{1} + \cdots + k\binom{n}{k} + \cdots + n\binom{n}{n}$$

as required.

(2) Show that $\binom{n}{0}^2 + \cdots + \binom{n}{k}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$ for any $n \geq 0$.

Solution: Consider the set $\{1, 2, \ldots, 2n\}$. The number of n-member subsets is $\binom{2n}{n}$.

On the other hand, for $k \leq n$, let S_k denote the number of n-member subsets of $\{1, \ldots, 2n\}$ k of whose members/elements belong to $\{1, \ldots, n\}$ (and the other $n-k$ belong to $\{n+1, \ldots, 2n\}$. For these subsets we have $\binom{n}{k}$ possibilities for the first k elements and $\binom{n}{n-k}$ for the remainder. This tells us that $S_k = \binom{n}{k}\binom{n}{n-k} = \binom{n}{k}\binom{n}{k} = \binom{n}{k}^2$.

Since $\binom{2n}{n} = S_0 + S_1 + \cdots + S_n$, the result follows.

(3) . Let p be a prime number.

Show that $p|\binom{p^m}{k}$ whenever $1 \leq k \leq p^m - 1$.

Solution: We use the following fundamental property of prime numbers:

If p is prime and if $p|ab$ but $p \not| a$ then $p|b$. (*)

Now, when $k < p^m$ we can write $k = p^s t$ where $p \not| t$ and $0 \leq s < m$. Then

$$\binom{p^m}{k} = \frac{p^m}{k}\binom{p^m - 1}{k-1} = \frac{p^{m-s}}{t}\binom{p^m - 1}{k-1}$$

where $m - s > 0$. Thus $t\binom{p^m}{k} = p^{m-s}\binom{p^m-1}{k-1}$ is an equation of products of integers. Since p divides the right-hand side it also divides the left-hand side. But $p \not| t$. By (*) with $a = t$, $b = \binom{p^m}{k}$, it follows that $p|\binom{p^m}{k}$.

(4) Let $N \geq 0$, $k \geq 1$ be integers. How many solutions are there to

$$x_1 + \cdots + x_k = N$$

with $x_1, \ldots, x_k$ non-negative integers?

Solution: As we saw, any solution can be encoded as a binary string of length $N + k - 1$ with $k - 1$ 1s. (The question is the same as: How amny ways can we distribute N sugar lumps in k cups: Let x_i be the number of sugar lumps in the ith cup.)

So the number of solutions is $\binom{N+k-1}{k-1} = \binom{N+k-1}{N}$.

(5) Let $N \geq 0$, $k \geq 1$ be integers. How many solutions are there to

$$x_1 + \cdots + x_k = N$$

where $x_1, \ldots, x_k$ are integers ≥ 1?

Solution: Let $y_i = x_i - 1$ for $i = 1, \ldots, k$. Then the equation can be re-written

$$y_1 + \cdots y_k = N - k$$

where the y_i are to be integers ≥ 0 (i.e., non-negative).

Thus the answer is: There are no solutions if $N < k$ (i.e. when $N - k < 0$), and $\binom{N-k+k-1}{k-1} = \binom{N-1}{k-1}$ otherwise.

(6) Let $N \geq 0$, $k \geq 1$ be integers. Let $r \geq 0$. How many solutions are there to

$$x_1 + \cdots + x_k = N$$

where $x_1, \ldots, x_k$ are integers $\geq r$?

Solution: Same idea as last problem: Let $y_i = x_i - r$. The equation becomes $y_1 + \cdots y_k = N - kr := M$ with $y_i \geq 0$.

So there are no solutions if $N < kr$ and otherwise there are $\binom{M+k-1}{k-1} = \binom{N+k(1-r)-1}{k-1}$ solutions.

(7) A bakery has 8 different types of doughnut. How many different 1-dozen boxes of doughnuts can it make?

Solution: Let x_i be the number of doughnuts of type i in a given box, $i = 1, 2, \ldots, 8$. So $x_i \geq 0$ and $x_1 + \cdots + x_8 = 12$.

Thus there are $\binom{12+7}{7} = \binom{19}{7}$ possible boxes.

(8) How many different non-decreasing sequences of length 7 can be made using only the numbers $1, 2, \ldots, 20$?

(Eg, one such sequence is $1, 2, 2, 11, 11, 11, 19$)

Solution: Let x_1 be the number of 1s occurring in a given sequence, x_2 the number of 2s ,etc .. x_{20} is the number of 20's. Since the sequence is non-decreasing, these numbers determine the sequence. Now $x_i \geq 0$ and $x_1 + \cdots x_{20} = 7$. So the number is $\binom{7+19}{19} = \binom{26}{7}$.