Interacting Particle Systems, Last Passage Percolation and Random Matrices

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Abstract

We study the Oriented Swap Process, and propose two conjectures relating to its absorbing time. The first conjecture states that the absorbing time has fluctuations of order $n^{1/3}$, where n is the size of the system, and converges to the GOE Tracy-Widom distribution. The second conjecture, which would be a first fundamental step to prove the first one, relates the absorbing time to a certain line-to-line last-passage percolation model. In support of our conjectures we present numerical evidence and basic computations.

1 The Oriented Swap Process

The *Oriented Swap Process* (OSP), introduced by Angel, Holroyd and Romik [1] is a continuous-time Markov process with state-space S_n, i.e. the symmetric group on n objects. The initial configuration of the model is the identity permutation, $\eta_0 = (1, \ldots, n)$. We will call each $i, 1 \leq i \leq n$, the i-th particle. Each particle has a clock that rings at random times; each time interval between two consecutive rings of a given clock is an exponential random variable of rate 1, and all of these variables are independent. Every time a particle's clock rings, it will attempt to swap position with the particle to its right. If they are in increasing order, they swap, otherwise, they do not. For example, if the clock of particle i rings first, then

$$(1, 2, \ldots, i, i+1, \ldots, n-1, n)$$

swaps to

$$(1, 2, \ldots, i+1, i, \ldots, n-1, n).$$

Define $\eta_t \in S_n$ to be the random configuration of the process at time t, and conversely $\eta_t^{-1}(k)$ to be the position of particle k at time t. The *absorbing time* or *total finishing time*, β_*^n of the process is the time when the model reaches the unique *absorbing state*

$$(n, n-1, \ldots, 2, 1).$$

When this state is reached, no more swaps are possible. The *finishing time*, $\beta^n(k)$, of particle k is the last time that it moves. Since particle k finishes in position $n-k+1$, we have that

$$\beta^n(k) := \sup \left\{ t > 0 : \eta_t^{-1}(k) \neq n-k+1 \right\}.$$

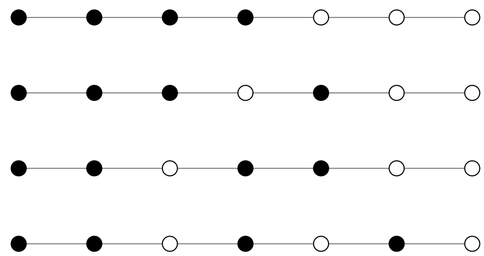


Figure 1: An example of the first three steps in the 4-th TASEP for $n = 7$.

The *absorbing time* is

$$\beta_*^n := \max_{k=1,\ldots,n} \beta^n(k).$$

In order to describe the OSP, Angel, Holroyd and Romik [1] used a mapping to the *Totally Asymmetric Simple Exclusion Process* (*TASEP*). The TASEP on a finite interval $[1, n] := \{1, 2, \ldots, n\} \subset \mathbb{Z}$ is a continuous-time Markov process on $\{0, 1\}^{[1,n]}$. The 1's represent 'particles' in this case, while the 0's represent 'holes'. Each particle is given a clock, where the time interval between two consecutive rings is an exponential random variable of rate 1. All of these variables are independent. When a particle's clock 'rings', it will attempt to 'jump' to the right, succeeding only if the position to the right is not occupied. The particles cannot leave the interval $[1, n]$, and eventually will reach their absorbing state, of the form $(0, \ldots, 0, 1, \ldots, 1)$.

The Oriented Swap Process can be mapped to a family of coupled TASEPs as follows. For $k \in [1, n]$, the k-th TASEP represents the positions of the first k particles in the Oriented Swap Process at any time:

$$T_t^{(k)}(x) := \begin{cases} 1 & \text{if } \eta_t(x) \leq k, \\ 0 & \text{if } \eta_t(x) > k. \end{cases} \tag{1}$$

Since $\eta_0 = (1, \ldots, n)$, the initial configuration of the k-th TASEP will be

$$\mathbf{1}_{\{i \leq k\}}.$$

We can thus find the position of any particle k in the OSP using the k-th and $(k-1)$-th TASEPs:

$$(T_t^{(k)} - T_t^{(k-1)})(x) = \begin{cases} 1 & \text{if } \eta_t(x) = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

See Figure 1 for a possible example of one of these TASEPs. We also define the *absorbing time of the k-th TASEP*, $V^n(k)$, as the last time that the k-th TASEP will change configuration:

$$V^n(k) = \inf\{t > 0 : T_t^{(k)} = \mathbf{1}_{[n-k+1,n]}\}. \tag{3}$$

We therefore have that

$$\beta^n(k) = V^n(k) \vee V^n(k-1).$$

It turns out that $V^n(k)$ can be expressed as a last-passage percolation time on an array W of exponential waiting times, as follows. The *Last-Passage Percolation* (LPP) time, $T(a,b;c,d)$, on an array $W=(w_{ij})_{i,j\in\mathbb{Z}}$ can be defined as:

$$T(a,b;c,d) := \max_{\pi:(a,b)\to(c,d)} \sum_{(i,j)\in\pi} w_{ij},$$

where π is any sequence $\pi = ((i_1,j_1),\ldots,(i_l,j_l))$ such that $(i_1,j_1)=(a,b)$, $(i_l,j_l)=(c,d)$ and $(i_{k+1},j_{k+1})-(i_k,j_k)\in\{(1,0),(0,1)\}$ for all $k=1,\ldots,l-1$. The length of π is $l=(d-b)+(c-a)+1$ i.e. the number of indices (i,j) in π. We can interpret π as a nearest neighbour path in $\mathbb{Z}$, that starts from (a,b), ends at (c,d), and moves only in the 'positive' directions $(0,1)$ and $(1,0)$ - it is therefore called *'oriented'*. It is well-known [3, 4] that if all $(w_{ij})_{i,j\in\mathbb{Z}}$ are i.i.d Exp(1) random variables, then:

$$V^n(k-1) \stackrel{d}{=} T(1,k;k-1,n). \tag{4}$$

2 Asymptotics of Finishing Times

Angel, Holroyd and Romik [1] proved laws of large numbers for the finishing times of individual particles in the 'bulk', and for the absorbing time of the OSP.

Theorem 1 (Laws of Large Numbers [1]). *Let $(k_n)_n$ be a sequence of integers such that $\frac{k_n}{n}\to y\in(0,1)$, as $n\to\infty$. Define $\gamma_y = 1+2\sqrt{y(1-y)}$. Then the following limits in probability hold:*

$$\begin{aligned} \frac{\beta^n(k)}{n} &\xrightarrow{\mathbb{P}} \gamma_y && \text{as } n\to\infty,\\ \frac{\beta^n_*}{n} &\xrightarrow{\mathbb{P}} \max_{y\in(0,1)}\gamma_y=\gamma_{1/2}=2 && \text{as } n\to\infty. \end{aligned}$$

This theorem describes the finishing times of fixed-ratio particles. For example, if $y=\frac{1}{2}$, then k would always be (one of) the middle particles (e.g. $k=50$ or 51 when $n=100$). Moreover, this result tells us that the absorbing time is asymptotically equal to the finishing time of the middle-most particle, that is $\gamma_{1/2}n=2n$. Angel, Holroyd and Romik [1] also proved the following 'central limit theorem' for the finishing times of the fixed ratio particles.

Theorem 2 (Limiting distribution of the finishing times [1]). *The following convergence in distribution holds:*

$$\frac{\beta^n(k)-\gamma_y n}{\gamma_y^{\frac{2}{3}}(y(1-y))^{-\frac{1}{6}}n^{\frac{1}{3}}} \Longrightarrow F_2,$$

where γ_y is defined as in Theorem 1 and F_2 is the Tracy-Widom GUE distribution.

Here, the GUE Tracy-Widom Distribution F_2 is the limiting distribution of the scaled maximum eigenvalue of the Gaussian Unitary Ensemble. This refers to Hermitian matrices with independent complex Gaussian entries

$$\begin{cases} A_{ii}=X_i & \text{for } 1\le i\le N,\\ A_{ij}=\overline{A_{ji}}=\frac{Y_{ij}+iZ_{ij}}{\sqrt{2}} & \text{for } 1\le i<j\le N, \end{cases} \tag{5}$$

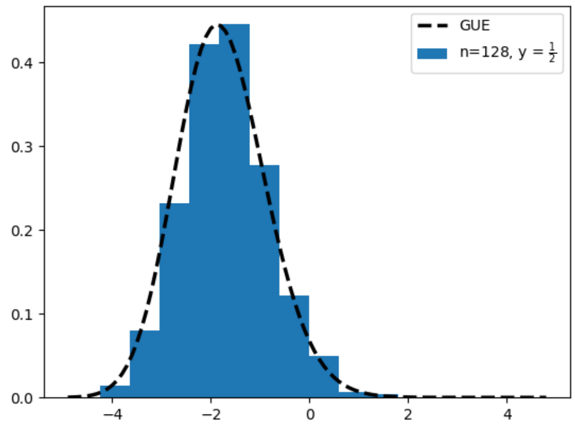


Figure 2: Tracy-Widom GUE distribution and scaled distribution of the finishing time of the middle particle in the OSP, with 2,140 data points.

where X_i, Y_{ij}, Z_{ij} are independent standard real Gaussian variables. In Figure 2, we have represented a histogram of the simulated finishing time of the middle particle, which, after rescaling, matches the GUE Tracy-Widom distribution. Theorem 2 has been proven using the coupling with TASEPs and LPPs, as explained in Section 1, along with a classical result of Johansson, i.e. Theorem 1.6 in [4]:

Theorem 3 (Johansson [4])**.** *Let* $T(1,1;M,N)$ *be the last-passage percolation time (see Section 1) from* $(1,1)$ *to* (M,N) *on an array of* $\mathrm{Exp}(1)$ *independent random variables. Then for* $\gamma \geq 1$*, we have the following convergence in distribution:*

$$\frac{T(1,1;\gamma n,n)-(1+\sqrt{\gamma})^2 n}{\gamma^{-\frac{1}{6}}(1+\sqrt{\gamma})^{\frac{4}{3}} n^{\frac{1}{3}}} \Longrightarrow F_2 \qquad \textit{as } n \to \infty. \tag{6}$$

As a consequence of the latter theorem, the absorbing times of both the k-th and $(k-1)$-th TASEP, $V^n(k)$ and $V^n(k-1)$, converge to the Tracy-Widom GUE distribution if $\frac{k}{n} \to y$ as $n \to \infty$. Angel, Holroyd and Romik deduced from this fact that also $\beta_n(k) = V^n(k) \vee V^n(k-1)$ has the same distributional limit under the same scaling. However, this method does not work to find the absorbing time β_*^n.
Indeed, Angel, Holroyd and Romik [1] stated an open question on the absorbing time β_*^n, which became the inspiration for this project. The following is an extract from Section 8 of [1]:

> *Limiting Distribution of the absorbing time.* Theorem 1.6 gives the limiting distribution of the fluctuations of the finishing times of individual particles. However, the relation between finishing times of different particles is more delicate and requires knowledge about the joint distribution of last-passage percolation times. An interesting open problem would be to find sequences

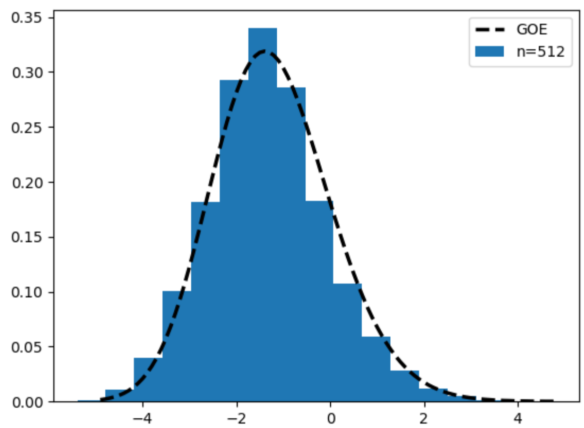


Figure 3: Tracy-Widom GOE Distribution and the scaled distribution of the absorbing time with 1,725 data points.

of scaling constants $(a_n)_n^\infty, (b_n)_n^\infty$ and a distribution function F such that the absorbing times of the oriented swap process satisfies the convergence in distribution

$$a_n(\beta_*^n - 2n) - b_n \Longrightarrow F \quad \text{ as } n \to \infty.$$

The rest of this report is a discussion of a_n's, b_n's and the nature of F. We will look to a Last-Passage Percolation model that jointly describes all finishing times of the Oriented Swap Process and, therefore, the absorbing time. This will be a first fundamental step to address the open problem above.

3 Numerics for the Absorbing Time

Our numerical investigations into the fluctuations of the absorbing time showed that they are of order $n^{\frac{1}{3}}$. This arose from plotting $\log(\sigma)$ against $\log(n)$, where σ is the standard deviation of the sample of finishing times, and $n \in \{128, 181, 256, 362\}$. The graph is shown in Figure 4. The slope of this graph tends to $\frac{1}{3}$ for sufficiently large n. Hence we conclude that:

$$\sigma \sim \sigma_* n^{\frac{1}{3}},$$

where σ_* is a constant as $n \to \infty$. Note here that the value for $\sigma_* \approx 1.575...$ is given by e^c, where c is the y-intercept of the line-of-best-fit in Figure 4. Remarkably, the numeric value for σ_* is less than the standard deviation

$$\sigma_* \approx 1.575... < \lim_{n\to\infty} \frac{\sigma(\beta^n(n/2))}{n^{1/3}} = \gamma_y^{\frac{2}{3}} (y(1-y))^{-\frac{1}{6}}\Big|_{y=\frac{1}{2}} \cdot \sigma(F_2) \approx 1.804...,$$

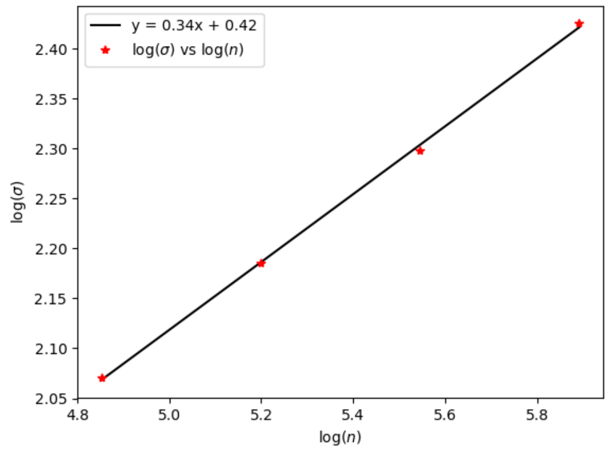


Figure 4: $\log(\sigma)$ vs $\log(n)$, with 92,985 data points.

where $\sigma(F_2) \approx 0.902...$ is the standard deviation of F_2 (see Theorem 2). This means that the distribution of the absorbing time is not a direct consequence of Theorem 2.

The order of the fluctuations suggests that the absorbing time is in the same universality class as that of the individual finishing times i.e. the Kardar-Parisi-Zhang Universality Class [3]. Therefore, we also expect the limiting distribution of the scaled absorbing time to arise from random matrix theory and, in particular, to be one of the Tracy-Widom distributions. We compared the histogram of the rescaled absorbing times with the probability density of the Tracy-Widom GUE, GOE and GSE distributions. The distribution matched that of the Tracy-Widom GOE distribution, as shown in Figure 3, and explained below.

The Tracy-Widom GOE Distribution F_1 is the limiting distribution of the scaled maximum eigenvalue of the Gaussian Orthogonal Ensemble. This refers to real symmetric matrices of size N with independent Gaussian entries

$$\begin{cases} A_{ii} = X_i & \text{for } 1 \leq i \leq N, \\ A_{ij} = \frac{Y_{ij}}{\sqrt{2}} & \text{for } 1 \leq i < j \leq N, \end{cases} \tag{7}$$

where X_i, Y_{ij} are independent standard real Gaussian variables.

For large n, we expect the absorbing time to be distributed about $2n$:

$$\beta_*^n \simeq 2n + cn^{1/3}\xi,$$

where ξ is a random variable in a distribution F and $c \in \mathbb{R}$. We therefore have that the standard deviation of the absorbing time is:

$$\sigma(\beta_*) = c\sigma(\xi)n^{\frac{1}{3}},$$

where $\sigma(\xi)$ is the standard deviation of F. Using our empirical value for the standard deviation, and our assumtion that F is the Tracy-Widom GOE, we have

$$c = \frac{\sigma_*}{\sigma(\xi)} = \frac{1.575...}{1.268...} \simeq 2^{\frac{1}{3}}.$$

This led us to formulate the following conjecture:

Conjecture 1 (Limiting distribution of the absorbing time)**.** *Let β_*^n be the absorbing time for the Oriented Swap Process with n particles. Then*

$$\frac{\beta_*^n - 2n}{(2n)^{\frac{1}{3}}} \overset{d}{\Longrightarrow} F_1, \qquad \text{as } n \to \infty, \tag{8}$$

where F_1 is the Tracy-Widom GOE distribution.

4 Line-to-Line Last-Passage Percolation Model

In this section we describe the absorbing time of the OSP as an upper-triangular array of waiting times, like

$$\begin{array}{ccc} w_{12} & w_{13} & w_{14} \\ & w_{23} & w_{24}, \\ & & w_{34} \end{array}$$

where the w_{ij}'s are independent exponential random times of rate 1. The relationship between the w_{ij}'s and the waiting times for the swaps in the Oriented Swap Process is unclear for now. In the case $n = 3$, we have a consistent definition for each w_{ij} in the following array:

$$\begin{array}{cc} w_{12} & w_{13} \\ & w_{23} \end{array}$$

We let w_{12} be the waiting time for the first successful swap of particles in positions 1 and 2, and w_{23} to be the waiting time for the first successful swap of particles in positions 2 and 3. Finally, we let w_{13} be the waiting time for the third swap, *after* the first two swaps have occurred. Figure 5 represents TASEPs for the case where the clock in position 1 rings before that in position 2. The waiting times between each jump are A, B and C. In this case, we have $w_{12} = A, w_{23} = A + B,$ and $w_{13} = C$. Thus, we have that $\beta_*^3 = w_{13} + w_{12} \vee w_{23}$. In fact, a stronger statement is true:

Proposition 1. *In the Oriented Swap Process for $n = 3$, we have the following equality of joint distributions:*

$$(V^3(1), V^3(2)) \overset{d}{=} (T(1,2;1,3), T(1,3;2,3)) = (w_{12} + w_{13}, w_{13} + w_{23}),$$

where $w_{12}, w_{13}, w_{23} \overset{iid}{\sim} \mathrm{Exp}(1)$. In particular, $\beta_*^3 \overset{d}{=} T(1,2;1,3) \vee T(1,3;2,3) = w_{13} + w_{12} \vee w_{23}$.

Proof. We want to show

$$\mathbb{P}(V^3(1) \leq s, V^3(2) \leq t) = \mathbb{P}(X + Y \leq s, X + Z \leq t),$$

where $X, Y, Z \overset{iid}{\sim} \mathrm{Exp}(1)$. By symmetry we may assume, without loss of generality, that $s < t$. The right-hand side is given by

$$\begin{aligned}
&\int_{\{x+y\le s,\, x+z\le t\}} e^{-x-y-z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad (9)\\
=&\int_0^{s\wedge t} e^{-x}\,\mathrm{d}x \int_0^{s-x} e^{-y}\,\mathrm{d}y \int_0^{t-x} e^{-z}\,\mathrm{d}z\\
=&\int_0^{s} e^{-x}\,\mathrm{d}x \int_0^{s-x} e^{-y}\,\mathrm{d}y \left(1-e^{-t+x}\right)\\
=&\int_0^{s} e^{-x}\left(1-e^{-t+x}\right)\mathrm{d}x \int_0^{s-x} e^{-y}\,\mathrm{d}y\\
=&\int_0^{s} e^{-x}\left(1-e^{-t+x}\right)\left(1-e^{-s+x}\right)\mathrm{d}x\\
=&\int_0^{s} e^{-x}-e^{-s}-e^{-t}+e^{-s-t+x}\\
=&1-e^{-s}-s(e^{-s}+e^{-t})+e^{-t}-e^{-s-t}.
\end{aligned}$$

To compute the left-hand side, let J be the waiting time for the first swap of the particles in positions 1 and 2. Let K be the waiting time for the first swap of particles in positions 2 and 3, and let M be the waiting time for the third swap *after* the first two swaps occur. Then the left-hand side is given by:

$$\begin{aligned}
&\mathbb{P}\left(V^3(1)\le s, V^3(2)\le t\right)\\
=&\mathbb{P}\left(V^3(1)\le s, V^3(2)\le t, J<K\right)+\mathbb{P}\left(V^3(1)\le s, V^3(2)\le t, K<J\right)
\end{aligned}$$

We deal first with the case where $J < K$, which is illustrated in Figure 5. In this case, J is the waiting time for the swap of particles 1 and 2, and K is the waiting time for the swap of particles 1 and 3. Since $M, K, J \overset{i.i.d}{=} \mathrm{Exp}(1)$, we have:

$$\begin{aligned}
&\mathbb{P}\left(V^3(1)\le s, V^3(2)\le t, J<K\right)\\
=&\mathbb{P}\left(K\le s, K+M\le t, J\le K\right)\\
=&\int_{\{0\le k\le s,\, 0\le k+m\le t,\, 0\le j\le k\}} e^{-k-j-m}\,\mathrm{d}k\,\mathrm{d}j\,\mathrm{d}m\\
=&\int_0^{s} e^{-k}\,\mathrm{d}k \int_0^{t-k} e^{-m}\,\mathrm{d}m \int_0^{k} e^{-j}\,\mathrm{d}j\\
=&\int_0^{s} e^{-k}\,\mathrm{d}k \int_0^{t-k} e^{-m}\,\mathrm{d}m(1-e^{-k})\\
=&\int_0^{s} e^{-k}(1-e^{-t+k})(1-e^{-k})\,\mathrm{d}k\\
=&\int_0^{s} e^{-k}-e^{-t}-e^{-2k}+e^{-k-t}\,\mathrm{d}k\\
=&1-e^{-s}-se^{-t}+\frac{e^{-2s}}{2}-\frac{1}{2}+e^{-t}-e^{-s-t}.
\end{aligned}$$

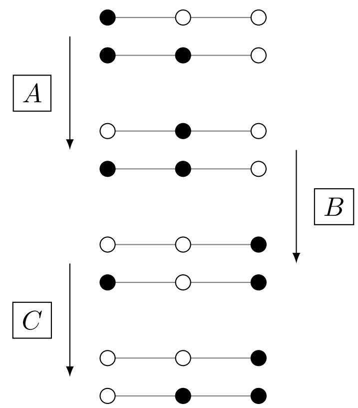


Figure 5: Coupled TASEPs for the OSP for $n = 3$, where the sequence of swaps is $(123) \to (213) \to (231) \to (321)$.

The second case, where $K < J$ is similar. With $s < t$, we have:

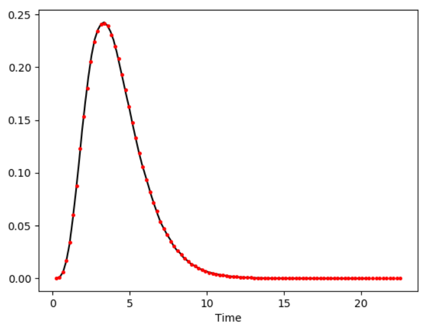
$$\begin{aligned}
&\mathbb{P}\left(V^3(1) \le s, V^3(2) \le t, K < J\right)\\
=&\mathbb{P}\left(J + M \le s, J \le t, K < J\right)\\
=&\int_{\{0 \le K < J,\, 0 \le J \le t,\, 0 \le M \le s-J\}} e^{-j-k-m}\,\mathrm{d}j\,\mathrm{d}m\,\mathrm{d}k\\
=&\int_0^s e^{-j}\,\mathrm{d}j \int_0^{s-j} e^{-m}\,\mathrm{d}m \int_0^j e^{-k}\,\mathrm{d}k\\
=&\int_0^s e^{-j}\,\mathrm{d}j \int_0^{s-j} e^{-m}(1 - e^{-j})\,\mathrm{d}m\\
=&\int_0^s e^{-j}(1 - e^{-s+j})(1 - e^{-j})\,\mathrm{d}j\\
=&\int_0^s e^{-j} - e^{-s} - e^{-2j} + e^{-s-j}\,\mathrm{d}j\\
=&1 - e^{-s} - se^{-s} + \frac{e^{-2s}}{2} - \frac{1}{2} + e^{-s} - e^{-2s}.
\end{aligned}$$

We therefore have

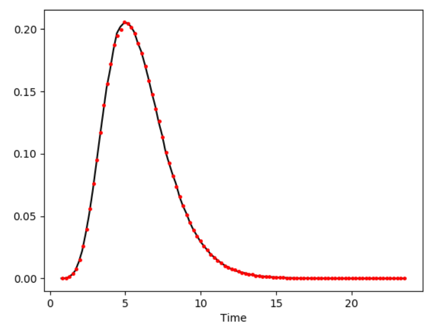
$$\mathbb{P}\left(V^3(1) \le s, V^3(2) \le t\right) = 1 - e^{-s} - s(e^{-s} + e^{-t}) + e^{-t} - e^{-s-t}$$

as required. □

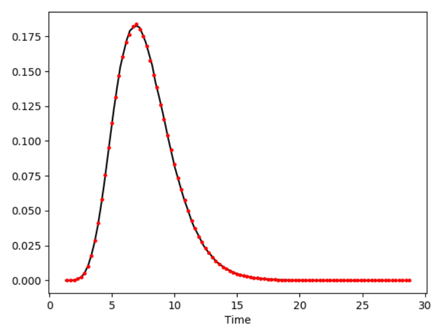
Looking to the cases where $n > 3$, we do not have consistent definitions for the w_{ij}'s. However, we conjecture that the absorbing time of the OSP for $n = 4$ is given by the maximum last-passage percolation time over all subarrays highlighted below from North-



(a) $n = 4$



(b) $n = 5$



(c) $n = 6$

Figure 6: The empirical distributions of β_*^n (black) and the Last-Passage time defined on the right-hand side of Equation (11) (red).

West (i.e. the top-left) to South-East (i.e. bottom right):

$$\begin{array}{ccc} \multicolumn{3}{|c|}{w_{12} \quad w_{13} \quad w_{14}} \\ & w_{23} & w_{24} \\ & & w_{34} \end{array} \qquad \begin{array}{ccc} w_{12} & \boxed{\begin{array}{cc} w_{13} & w_{14} \\ w_{23} & w_{24} \end{array}} \\ & \quad\;\; w_{34} \end{array} \qquad \begin{array}{cc} w_{12} \;\; w_{13} & \boxed{\begin{array}{c} w_{14} \\ w_{24} \\ w_{34} \end{array}} \end{array}$$

We conjecture that a similar statement is true for all n. Figure 6 provides numerical evidence of the identity in distribution between the absorbing time and the last-passage percolation time for the cases $n = 4, 5, 6$. In analogy with the case $n = 3$ worked out in Proposition 1, we actually conjecture a stronger statement:

Conjecture 2 (Line-to-Line Last-Passage Percolation)**.** *Let $W = (w_{ij})_{1 \le i < j \le n}$ be an array of i.i.d* Exp(1) *waiting times. We have the equality in distribution*

$$(V^n(1), V^n(2), \ldots, V^n(n-1)) \stackrel{d}{=} (T(1,2;1,n), T(1,3;2,n), \ldots, T(1,n;n-1,n)). \quad (10)$$

where $T(1,k;k-1,n)$ is the last passage time from $(1,k)$ to $(k-1,n)$ in the subarray

$W[1,k;k-1,n]$. *In particular:*

$$\beta_*^n \stackrel{d}{=} \max_{k=2,\ldots,n} T(1,k;k-1,n), \tag{11}$$

Note that the marginals of the right-hand side of Equation 10 are known to be equivalent, from Equation 4 and [3, 4]. We call the variable on the right-hand side of Equation 11 "Line-to-Line Last-Passage Percolation", because it is taken over all oriented paths of length $n-1$ from points on L_1 to points on L_2, where, in the array W above:

$$L_1 = \{(1,k) : k \in \mathbb{Z}\},$$
$$L_2 = \{(k,n) : k \in \mathbb{Z}\}.$$

Using the Robinson-Schensted-Knuth (RSK) correspondence and its variations, in a similar fashion to [2, 4], we should be able to prove that the above line-to-line model is equal in distribution to the point-to-line model in the case of exponential random variables. The point-to-line last-passage percolation is given by

$$\tau_n := \max_{i+j=n} T(1,1;i,j) \tag{12}$$

on an array $(w_{i,j})_{i,j\geq 1, i+j\leq n}$. If this equality were proven, along with Conjecture 2 above, then Theorem 4 below, from [2], would lead to a proof of Conjecture 1.

Theorem 4 (Limit Law of Point-to-Line Last-Passage Percolation [2])**.** *If the waiting times are independent and exponentially distributed with rate 1, then the following convergence in distribution holds:*

$$\frac{\tau_n - 2n}{(2n)^{\frac{1}{3}}} \Longrightarrow F_1 \qquad \text{as } n \to \infty. \tag{13}$$

where τ_n is the point-to-line last-passage percolation time, and F_1 is the Tracy-Widom GOE distribution.

References

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