THE PRINCIPLE OF INDUCTION

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The Principle of Induction: Let a be an integer, and let $P(n)$ be a statement (or proposition) about n for each integer $n \geq a$. The principle of induction is a way of proving that $P(n)$ is true for all integers $n \geq a$. It works in two steps:

(a) [**Base case:**] Prove that $P(a)$ is true.
(b) [**Inductive step:**] Assume that $P(k)$ is true for some integer $k \geq a$, and use this to prove that $P(k+1)$ is true.

Then we may conclude that $P(n)$ is true for all integers $n \geq a$.

This principle is very useful in problem solving, especially when we observe a *pattern* and want to prove it.

The trick to using the Principle of Induction properly is to spot *how to use* $P(k)$ to prove $P(k+1)$. Sometimes this must be done rather ingeniously!

Problem 1. Prove that for any integer $n \geq 1$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 .$$

Solution. Let $P(n)$ denote the proposition to be proved. First let's examine $P(1)$: this states that

$$1^3 = \left[\frac{1(2)}{2}\right]^2 = 1^2 ,$$

i.e., $1 = 1$, which is correct.

Next, we assume that $P(k)$ is true for some positive integer k, i.e.

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 .$$

We want to use this to prove $P(k+1)$, i.e.

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2 .$$

Taking the LHS and using $P(k)$,

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 &= \left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + (k+1)^3 \\
&= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)(k+1)^2}{4} \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left[\frac{(k+1)(k+2)}{2}\right]^2 .
\end{aligned}$$

and thus $P(k+1)$ is true. This completes the proof.

Note that we used $P(k)$ in the second line of the above calculations – this is very important and is the *key step* to making the solution easier than it would be without induction.

Problem 2. A sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and

$$a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \cdots + a_{n-1}) \text{ for all } n \geq 2 \, .$$

Determine a_{100}.

Solution. Using the recursive formula, we can generate the first few values of a_n, and tabulate these:

n	a_n	$n+1$	$\frac{a_n}{n+1}$	2^n	2^{n-2}
1	1	2	1/2	2	1/2
2	3	3	1	4	1
3	8	4	2	8	2
4	20	5	4	16	4
5	48	6	8	32	8

Based on this, we make the conjecture that

$$a_n = (n+1)2^{n-2} \text{ for all } n \geq 1 \, .$$

Call this claim $P(n)$. We will prove this claim by induction. First, we check the base case $P(1)$:

$$a_1 = 2 \cdot 2^{-1} = 1 \, ,$$

which is true.

Next, assume that $P(k)$ is true, i.e.,

$$a_k = \left(\frac{k+1}{k-1}\right)(a_1 + a_2 + \cdots + a_{k-1}) = (k+1)2^{k-2} \, .$$

We wish to use this hypothesis to prove that $P(k+1)$ is true, i.e., that

$$a_{k+1} = \left(\frac{k+2}{k}\right)(a_1 + a_2 + \cdots + a_k) = (k+2)2^{k-1} \ .$$

To achieve this, consider

$$\begin{aligned}
a_{k+1} &= \left(\frac{k+2}{k}\right)(a_1 + a_2 + \cdots + a_k) \\
&= \left(\frac{k+2}{k}\right)[(a_1 + a_2 + \cdots + a_{k-1}) + a_k] \\
&= \left(\frac{k+2}{k}\right)\left[\left(\frac{k-1}{k+1}\right)a_k + a_k\right] \\
&= \left(\frac{k+2}{k}\right)\left(\frac{2k}{k+1}\right)a_k \\
&= 2\left(\frac{k+2}{k+1}\right)a_k \\
&= 2\left(\frac{k+2}{k+1}\right)(k+1)2^{k-2} \\
&= (k+2)2^{k-1}
\end{aligned}$$

and this proves $P(k+1)$. Note that we used $P(k)$ in the sixth line of the calculation above.

Thus we conclude by the principle of induction that

$$a_n = (n+1)2^{n-2} \text{ for all } n \geq 1 \ .$$

It follows that $a_{100} = 101 \cdot 2^{98}$.

Problem 3. $2n$ points are given in space, where $n \geq 2$. Altogether n^2+1 line segments ('edges') are drawn between these points. Show that there is at least one set of three points which are joined pairwise by line segments (i.e. show that there exists a *triangle*).

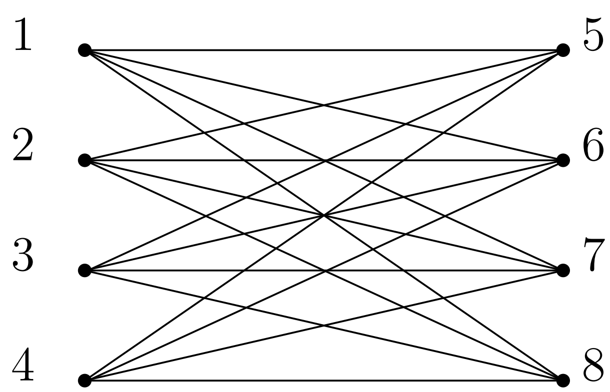
Solution. The proposition (let's call it $P(n)$) holds for $n = 2$ (why?). Assume that the proposition $P(n)$ is true for $n = k$, i.e. that if $2k$ points are joined together by $k^2 + 1$ edges, there must exist a triangle.

Now consider $P(k + 1)$: here we have $2(k + 1) = 2k + 2$ points, which are connected by $(k + 1)^2 + 1 = k^2 + 2k + 2$ edges.

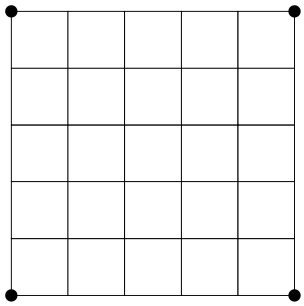
Take a pair of points A, B which are joined by an edge (there must be such a pair, otherwise there are no edges connecting any of the points!). The remaining $2k$ points form a set which we will call $\mathcal{S}$. Let's focus on the set $\mathcal{S}$ for the moment. If there were at least k^2+1 edges in $\mathcal{S}$, then there would have to be a triangle in here (using the $P(k)$ assumption). Of course there could be $\leq k^2$ edges in $\mathcal{S}$; let's suppose this is the case. But if this were true, it would mean that there are at least $2k + 2$ edges in the other part of the graph, i.e. connecting A and B to each other and to the points in $\mathcal{S}$. Discounting the edge AB gives at least $2k+1$ edges which connect from A or B into $\mathcal{S}$. But we notice that if P is a point in $\mathcal{S}$, then P can be connected either to A or B, but not both (or a triangle

PAB would be formed!). Therefore the maximum number of edges connecting from A or B into $\mathcal{S}$ (without forming a triangle) is $2k$. This contradiction proves that $P(k+1)$ must be true.

Note. If we have $2n$ points and *exactly* n^2 edges, it is possible to *avoid* making a triangle. This is done by breaking the set of points into two subsets $\mathcal{X}$ and $\mathcal{Y}$ which contain n points each, then connecting every point in $\mathcal{X}$ to every point in $\mathcal{Y}$. This is illustrated below for the case $n = 4$.



Problem 4 (from IrMO 2012). Let n be a positive integer. A mouse sits at each corner point of an $n \times n$ square board, which is divided into unit squares as shown below for the example $n = 5$.



The mice then move according to a sequence of *steps*, in the following manner:

(a) In each step, each of the four mice travels a distance of one unit in a horizontal or vertical direction. Each unit distance is called an *edge* of the board, and we say that each mouse *uses* an edge of the board.

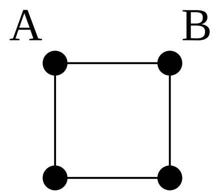
(b) An edge of the board may not be used twice in the same direction;

(c) At most two mice may occupy the same point on the board at any time.

The mice wish to collectively organise their movements so that each edge of the board will be used twice (not necessarily by the same mouse), and each mouse will finish up at its starting point. Determine, with proof, the values of n for which the mice may achieve this goal.

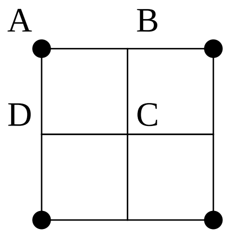
Solution. We prove by induction that the mice may achieve this goal for every positive integer n. We will focus on the movements of one mouse starting at one corner A of the board.

First, for $n = 1$, consider the following figure:



The mouse moves $A \to B \to A$. Successive mice move in a symmetrical fashion (rotated by 90 degrees).

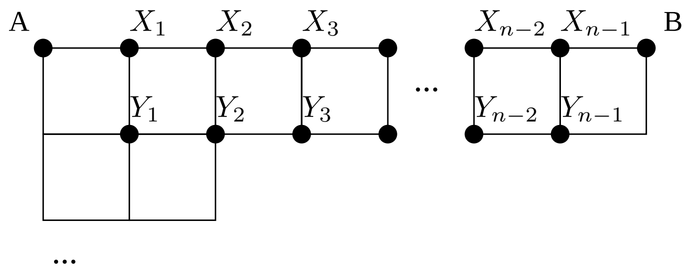
Next, for $n = 2$, consider the following figure:



Two mice move $A \to B \to C \to B \to A \to D \to A$, and the other two move $A \to D \to A \to B \to C \to B \to A$ (with the understanding that the moves for each successive mouse are rotated by 90 degrees). Note that at four points during this process, two mice occupy the same point of the board at the same time.

Next, for $n > 2$, we assume that a solution exists for an $(n-2) \times (n-2)$ board.

Consider the figure below.



The mouse starting at A can execute the moves:

$$A \to X_1 \to Y_1 \to X_1$$

$$\to X_2 \to Y_2 \to X_2$$

$$\to X_3 \to Y_3 \to X_3$$

$$\cdots \to X_{n-1} \to Y_{n-1} \to X_{n-1}$$

$$\to B$$

$$\to X_{n-1} \to X_{n-2} \to X_{n-3} \cdots \to X_1 \to A\ .$$

If the other four mice execute the same moves rotated by 90 degrees, then

(a) Together the mice will use every edge exactly once in each direction, apart from the inner $(n-2) \times (n-2)$ square;

(b) When the mouse starting from A reaches position Y_1, all four mice will be at the corners of the inner $(n-2) \times (n-2)$ square; thus the solution for $n-2$ can be spliced in at this moment before the mice continue on their homeward journey.

It follows by the principle of induction that a solution can be achieved for every positive integer n.

Exercise 1. Prove that for any integer $n \geq 1$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} .$$

Exercise 2. Show that for all integers $n \geq 1$,

$$1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{n(4n^2-1)}{3} .$$

Exercise 3.

Let m and n be positive integers. Given a $(2m+1) \times (2n+1)$ chessboard in which all four corners are black squares, show that if one removes any one white square and any two black squares, the remaining board can be covered with non-overlapping dominoes (here, a *domino* is a 1×2 rectangle).

Exercise 4.

Let $f(n)$ be the number of regions which are formed by n lines in the plane, where no two lines are parallel and no three meet at a single point (e.g. $f(1) = 2$; $f(2) = 4$; etc.). Find a formula for $f(n)$.

Exercise 5. Every road in Uniland is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly of via at most one other city.

Exercise 6. A row of n lamps is labelled from left to right with the numbers 1 to n, where n is an *odd* positive integer. Each lamp has a switch which, if pressed, turns it from OFF to ON or from ON to OFF; however the switches may only be pressed according to the following rules:

(a) Switch 1 may be pressed at any time;

(b) Switch $k \in \{2, \ldots, n\}$ may be pressed if and only if lamp $(k-1)$ is ON and all lamps $\ell < (k-1)$ (if any) are OFF.

Initially all lamps are ON. Prove that the minimum number of times the switches must be pressed to turn all the lamps OFF is

$$\frac{2^{n+1} - 1}{3}.$$

Pólya's Paradox:

A common way (in 1950, at least!) of expressing that something is out of the ordinary is "*That's* a horse of a different color!" The famous mathematician George Pólya gave the following proof that "all horses are the same color", which works by the principle of induction:

Proposition $P(n)$: Suppose we have n horses. Then all n horses are the same colour.

Base case: $n = 1$; if there is only one horse, there is only one colour.

Inductive step: Assume that $P(k)$ is true, i.e. that for any set of k horses, there is only one color. Now look at any set of $k + 1$ horses; call this $\{H_1, H_2, H_3, \cdots, H_k, H_{k+1}\}$. Consider the sets $\{H_1, H_2, H_3, \cdots, H_k\}$ and $\{H_2, H_3, H_4, ..., H_{k+1}\}$. Each is a set of only k horses, therefore within each there is only one colour. But the two sets overlap, so there must be only one colour among all $k + 1$ horses.

The flaw is that when $k = 2$ the inductive step doesn't work, because the statement that "the two sets overlap" is false.