Inequalities

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The Arithmetic Mean – Geometric Mean (AM-GM) Inequality:

For any two positive real numbers x and y, we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

with equality if and only if $x = y$.

The quantity of the LHS is called the *arithmetic mean* of the two numbers x and y. The quantity of the LHS is called the *geometric mean* of the two numbers x and y. They can be regarded as providing two different ways of "averaging" a pair of numbers.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of two positive quantities whose *product* is fixed occurs when both are equal.
- The *maximum* value of the *product* of two positive quantities whose *sum* is fixed occurs when both are equal.
- A geometric interpretation of this result is that in any right-angled triangle, the median corresponding to the hypotenuse is bigger than the altitude corresponding to hypothenuse.

Example. Find the minimum of $x + \frac{5}{x}$, where x is positive.

Solution. By the AM-GM inequality,

$$\begin{aligned} x + \frac{5}{x} &\geq 2\sqrt{(x) \cdot \left(\frac{5}{x}\right)} \\ &= 2\sqrt{5}\,. \end{aligned}$$

The minimum occurs when $x = \frac{5}{x}$, i.e., when $x = \sqrt{5}$.

Example. Prove that for any positive numbers a, b and c we have

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. By the AM-GM inequality we have

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \frac{b+c}{2} \geq \sqrt{bc}, \quad \frac{c+a}{2} \geq \sqrt{ca}$$

If we multiply these three inequalities we find

$$\frac{(a+b)(b+c)(c+a)}{8} \geq \sqrt{(ab)(bc)(ca)} = abc$$

and this finishes our proof.

The general AM-GM Inequality

Suppose we have n positive real numbers $x_1, x_2, \ldots, x_n$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Example.

Minimize $x^2 + y^2 + z^2$ subject to $x, y, z > 0$ and $xyz = 1$.

Solution. By AM-GM,

$$\begin{aligned} x^2 + y^2 + z^2 &\geq 3\sqrt[3]{x^2 \cdot y^2 \cdot z^2} \\ &= \sqrt[3]{(xyz)^2} \\ &= 1\ . \end{aligned}$$

The minimum occurs when $x^2 = y^2 = z^2$, i.e., when $x = y = z = 1$.

Example.

Minimize $\frac{6x}{y}+\frac{12y}{z}+\frac{3z}{x}$ for $x, y, z > 0$.

Solution. By AM-GM,

$$\frac{6x}{y}+\frac{12y}{z}+\frac{3z}{x} \geq 3\sqrt[3]{\frac{6x}{y}\cdot\frac{12y}{z}\cdot\frac{3z}{x}} = 3\sqrt[3]{6\cdot 12\cdot 3} = 3\cdot 6 = 18\ .$$

The minimum occurs if and only if $\frac{6x}{y}=\frac{12y}{z}=\frac{3z}{x}$, i.e., if and only if $x=t$, $y=t$ and $z=2t$ for some positive number t.

Example.

Let x be a positive number. Minimize $x^2+\frac{6}{x}$.

Solution. We seek to minimize the sum of two quantities. Note that the product of the two quantities is equal to $6x$ – this is NOT a constant. However, we can rearrange the sum as

$$x^2+\frac{3}{x}+\frac{3}{x}\ .$$

and the product of the three terms is 9.

Using AM-GM inequality we find

$$x^2+\frac{3}{x}+\frac{3}{x} \geq 3\sqrt[3]{x^2\cdot\frac{3}{x}\cdot\frac{3}{x}} = 3\sqrt[3]{9}\ .$$

The equality occurs when $x^2=\frac{3}{x}$, i.e. when $x=\sqrt[3]{3}$.

Two More "Averages":

The **Harmonic Mean** of n numbers $x_1, x_2, \ldots, x_n$ is given by

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

and their **Root-Mean-Square** is given by

$$\mathrm{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}\ .$$

If all the numbers $x_1, x_2, \ldots, x_n$ are positive, then we have

$$\min\{x_1, \ldots, x_n\} \leq \mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM} \leq \mathrm{RMS} \leq \max\{x_1, \ldots, x_n\}$$

with equality in each case if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Special case: for two positive numbers x and y

$$\min\{x, y\} \leq \frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}} \leq \max\{x, y\}\ .$$

Exercise: Prove the above special case (*all* inequalities)!

Looking at the AM-HM inequality, we have AM $\geq$ HM, or

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

This can be rearranged into the form

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2 \ ,$$

with equality if and only if the numbers $x_1, x_2, \ldots, x_n$ are all equal.

Example: "Nesbitt's Inequality".

Prove that for positive numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2} \ .$$

Solution. Write the LHS as

$$\begin{aligned}
&\frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3 \\
&\qquad = (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3 \\
&\qquad = \frac{1}{2}\left[(a+b) + (b+c) + (a+c)\right]\left[\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right] - 3 \\
&\qquad \geq \frac{1}{2}(9) - 3 = \frac{3}{2}
\end{aligned}$$

where we have used the HM-AM inequality with $n = 3$:

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3^2$$

with $x = a + b$, $y = b + c$, $z = a + c$.

Example (Selection test 2016)

Prove that for any positive real numbers a, b and c we have

$$\frac{2a+b}{b+2c}+\frac{2b+c}{c+2a}+\frac{2c+a}{a+2b}\geq 3.$$

Solution. Let

$$b+2c=x \qquad (1)$$

$$c+2a=y \qquad (2)$$

$$a+2b=z \qquad (3)$$

Adding the above equalities we find

$$2a+2b+2c=\frac{2(x+y+z)}{3} \qquad (4)$$

Now, from (1) and (4) we find

$$2a+b=\frac{2y+2z-x}{3}$$

and similarly,

$$2b+c=\frac{2x+2z-y}{3} \qquad \text{and} \qquad 2c+a=\frac{2x+2y-z}{3}.$$

Thus, in the new variables x, y, z our initial inequality reads

$$\frac{1}{3}\left\{\frac{2y+2z-x}{x}+\frac{2x+2z-y}{y}+\frac{2x+2y-z}{z}\right\}\geq 3,$$

or even

$$2\Big(\frac{x}{y}+\frac{y}{x}\Big)+2\Big(\frac{y}{z}+\frac{z}{y}\Big)+2\Big(\frac{x}{z}+\frac{z}{x}\Big)\geq 12. \qquad (5)$$

By AM-GM inequality we have

$$\frac{x}{y} + \frac{y}{x} \geq 2, \quad \frac{y}{z} + \frac{z}{y} \geq 2, \quad \frac{x}{z} + \frac{z}{x} \geq 2.$$

Adding the above inequalities we find (5) which proves our initial inequality.

Example. Let $a, b, c > 0$ be such that $abc = 1$. Prove that

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} \geq 3.$$

Solution. Observe first that

$$\frac{1+ab}{1+a} = \frac{abc+ab}{1+a} = ab\frac{1+c}{1+a}.$$

Similarly,

$$\frac{1+bc}{1+b} = bc\frac{1+a}{1+b}, \quad \frac{1+ca}{1+c} = ca\frac{1+b}{1+c}.$$

By AM-GM inequality we now obtain

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c} = ab\frac{1+c}{1+a} + bc\frac{1+a}{1+b} + ca\frac{1+b}{1+c}$$

$$\geq 3\sqrt[3]{ab\frac{1+c}{1+a} \cdot bc\frac{1+a}{1+b} \cdot ca\frac{1+b}{1+c}} = 3\sqrt[3]{(abc)^2} = 3.$$

Sometimes we can be asked to prove an inequality regarding the *sides lengths of a triangle*. Here, the side lengths a, b, c (aside from being positive) must satisfy the so-called *triangle inequalities*:

$$a + b > c\,; \quad b + c > a\,; \quad c + a > b\,;$$

Example.

Let a, b, c be the side lengths of a triangle. Prove that

$$a^2 + b^2 + c^2 < 2(ab + bc + ca).$$

Solution.

Note that for example, if $a = 5$ and $b = c = 1$, we have

$$a^2 + b^2 + c^2 = 27\,; \quad 2(ab + bc + ca) = 22.$$

and the result does *not* hold. Therefore, it is important that we use the information that a, b, c satisfy the triangle inequalities.

Writing the triangle inequality $a + b > c$ as $c - b < a$ and squaring, we obtain $(c - b)^2 < a^2$. Doing this for each triangle inequality yields

$$(c - b)^2 < a^2$$

$$(a - b)^2 < c^2$$

$$(c - a)^2 < b^2$$

Adding these three inequalities, and simplifying, yields the result (**Exercise:** check this!).

Example.

Let a, b, c be the side lengths of a triangle. Prove that

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Solution. Since $a, b, c,$ are the sides of a triangle we have $a + b > c$, $b + c > a$ and $c + a > b$ which shows that the brackets on the right-hand side in the above inequality are also positive numbers.

There exists $x, y, z > 0$ such that

$$a = x + y, \quad b = y + z, \quad c = z + x.$$

Then, the above inequality is equivalent to

$$(x+y)(y+z)(z+x) \geq 8xyz.$$

But this follows now in a standard way by using

$$x + y \geq 2\sqrt{xy}, \quad y + z \geq 2\sqrt{yz}, \quad z + x \geq 2\sqrt{zx}.$$

Exercises.

(1) Let $x, y > 0$. Find the minimum of
$$\frac{50}{x} + \frac{20}{y} + xy.$$

(2) If $x > y > 0$, find the minimum of
$$x + \frac{8}{y(x-y)}.$$

(3) Prove that for any positive real numbers a, b, c we have
$$(a+9b)(b+9c)(c+9a) \geq 216abc.$$

(4) Prove that for positive real numbers x, y, z,
$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$
and determine when equality occurs.

(5) Find the positive number whose square exceeds its cube by the greatest amount.

(6) Prove that for positive real numbers x, y, z,
$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

For further reading, click here: Wikipedia entry on AM-GM