UCD Mathematics Enrichment Programme 2013

Two lectures on INEQUALITIES by

T. J. LAFFEY

Lecture 1. Saturday February 2.

$\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}$ can be partitioned into three subsets (i) the positive real numbers, P say, (ii) $\{0\}$ and (iii) the negative real numbers. A real number r is negative if and only if $-r$ is a positive real number.

Suppose a, b are positive real numbers. Then $a+b$ is positive and ab is positive. For $a, b \in \mathbb{R}$,

$a > 0$ means a is positive

$a \geq 0$ means $a = 0$ or a is positive.

$a > b$ means $a - b > 0$

$a \geq b$ means $a = b$ or $a - b > 0$.

Also $a > b$ is the same as $b < a$.

$a \geq b$ is the same as $b \leq a$.

Properties (1) If $a, b, c \in \mathbb{R}$ with $a > b$ and $c > 0$, then $ac > bc$.

Proof $a > b$ implies $a - b > 0$, so, since $c > 0$, also $(a-b)c > 0$, that is $ac - bc > 0$, so $ac > bc$

(2) If $a, b, c \in \mathbb{R}$ with $a > b$, then $a+c > b+c$.

Proof $a > b$ means $a - b > 0$. But $(a+c) - (b+c) = a - b$ (since $c, -c$ cancel). So $a+c > b+c$.

(3) If $a, b \in \mathbb{R}$ with $a > b > 0$. Then $a^2 > b^2$.

Proof $a^2 - b^2 = (a-b)(a+b)$. Now $a > b$ implies $a - b > 0$ while $a > b > 0$ implies $a+b > 0$. So $a^2 - b^2 > 0$.

(4) More generally, if $a > b > 0$ and k is a positive integer, then $a^k > b^k$.

Proof $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + ab^{k-2} + b^{k-1})$

(to prove this, just multiply out the right-hand-side). Now $a > b$ so $a - b > 0$ and since $a > 0$, $b > 0$, the second factor is positive. So $a^k - b^k > 0$ and $a^k < b^k$.

(5) If $a, b, c \in \mathbb{R}$ with $a > b$ and $c \underline{<} 0$. Then $ac \underline{<} bc$.

Proof. $a > b$ implies $a - b > 0$ and $c < 0$ implies $-c > 0$. Hence $(a-b)(-c) > 0$, that is $-ac + bc > 0$. So $ac < bc$.

(6) If $a > b > 0$ and k is a positive integer, then $a^{1/k} > b^{1/k}$.

Proof. First $a^{1/k} \neq b^{1/k}$ since $a \neq b$. Next, suppose $b^{1/k} > a^{1/k}$. Since $a^{1/k} > 0$ and $b^{1/k} > 0$ we can apply (4) with a, b replaced by $b^{1/k}, a^{1/k}$, respectively, to get $b = (b^{1/k})^k > (a^{1/k})^k = a$, which is false. Hence $b^{1/k}$ is not greater than $a^{1/k}$. Thus $a^{1/k} > b^{1/k}$.

Question. Suppose $S > 0$ and $a > 0$, $b > 0$ are real numbers with $a + b = S$. How large (in terms of S) could ab be.

Solution. Suppose $a > b$. Then $a > \frac{S}{2} > b$. Let x be a real number with $0 < x < \frac{S}{2} - b$ and let $b_0 = b + x$ and $a_0 = a - x$. Note that $b_0 < \frac{S}{2}$ and $a_0 + b_0 = (b+x) + (a-x) = S$, so $a_0 + b_0 = S$. Also $a_0 > 0$, $b_0 > 0$. Now $a_0 b_0 = (a-x)(b+x) = ab + x(a-b) - x^2 = ab + (a-b-x)x$. But $a > \frac{S}{2} > b + x$, so $a - b - x > 0$. So $a_0 b_0 > ab$. So if $a > b$, the product is not greatest possible (as a_0, b_0

satisfy the conditions and have a bigger product.
A similar argument works if $b > a$. So to get the biggest possible product, we must take $a = b = S/2$.
So the biggest possible product is $\left(\frac{S}{2}\right)^2 = \frac{S^2}{4} = \frac{(a+b)^2}{4}$.

Now if $a > 0$, $b > 0$ are real numbers, we have $ab \le \left(\frac{a+b}{2}\right)^2$, as the previous argument shows that the product of two positive real numbers whose sum is $S = a+b$ cannot exceed $\left(\frac{S}{2}\right)^2 = \frac{(a+b)^2}{4}$.
Alternatively, $\frac{(a+b)^2}{4} - ab = \frac{a^2 + 2ab + b^2 - 4ab}{4}$
$= \frac{(a-b)^2}{4} \geqslant 0$ (with equality only for $a = b$).
Using Property (6) we get: if $a > 0$ and $b > 0$, then $\sqrt{ab} \le \frac{a+b}{2}$ with equality only if $a = b$.

<u>New Question</u>: Suppose S is a positive real number and a, b, c are positive real numbers with $a+b+c = S$. What is the biggest possible value of abc?
<u>Solution</u>: Suppose we have chosen positive numbers a, b, c with $a+b+c = S$ and abc greatest possible subject to those conditions. If $a > b$, let $a_0 = \frac{a+b}{2}$, $b_0 = \frac{a+b}{2}$.
Then a_0, b_0, c are positive with $a_0 + b_0 + c = S$ and since $a > b$, $a_0 b_0 > ab$ and $a_0 b_0 c > abc$. So the fact that abc is greatest possible is contradicted. Hence a is not greater than b. A similar contradiction arises if $b > a$ or in general if any two of a, b, c are unequal. So the maximum possible product arises when $a = b = c = S/3$ and the maximum possible product is $\left(\frac{a+b+c}{3}\right)^3 = \frac{S^3}{27}$.

This proves:

<u>Proposition</u> Suppose a, b, c are positive real numbers. Then $abc \leq \left(\frac{a+b+c}{3}\right)^3$.

<u>Proof</u> Put $S = a+b+c$ and use the last result.

We have the following general result:

<u>AGM (Arithmetic - Geometric Mean Inequality)</u>.

Let $x_1, x_2, \ldots, x_n$ be <u>positive real numbers</u>. Then $x_1 x_2 \cdots x_n \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)^n$

(Equivalently

Geometric mean $\rightarrow$ $\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$.) Arithmetic Mean

We get equality precisely when $x_1 = x_2 = \cdots = x_n$.

<u>Proof</u>. Let $S = x_1 + x_2 + \cdots + x_n$ and choose $y_1 > 0$, $y_2 > 0$, $\ldots$, $y_n > 0$ with $y_1 + y_2 + \cdots + y_n = S$ and $y_1 y_2 \cdots y_n$ <u>greatest possible subject to</u> this. If $y_1 < y_2$, replace y_1, y_2 by $y_1' = y_2' = \frac{y_1 + y_2}{2}$. Notice that $y_1' > 0, y_2' > 0, y_3 > 0, \ldots, y_n > 0$ and $y_1' + y_2' + y_3 + \cdots + y_n = S$ and $y_1' y_2' > y_1 y_2$, so $y_1' y_2' y_3 \cdots y_n > y_1 y_2 y_3 \cdots y_n$ which contradicts our choice of $y_1, \ldots, y_n$. Hence $y_1 < y_2$ is not possible. The same type of argument shows that $y_2 < y_1$ is impossible. So $y_1 = y_2$ and similarly $y_1 = y_3$, $y_1 = y_4$, $\ldots$, $y_1 = y_n$ and each $y_i = \frac{S}{n}$ and $y_1 y_2 \cdots y_n = \left(\frac{S}{n}\right)^n$. It follows that $x_1 x_2 \cdots x_n \leq \left(\frac{S}{n}\right)^n$ with equality only when all the xs are equal.

Example $2 \times 3 \times 4 \times 5 \leq \left(\frac{2+3+4+5}{4}\right)^4 = \left(\frac{14}{4}\right)^4 = \left(\frac{7}{2}\right)^4$

so $\sqrt[4]{120} < \frac{7}{2}$.

$2 \times 3 \times 4 \times 5 = 120$ and $\left(\frac{7}{2}\right)^4 = \frac{2401}{16} = 150\frac{1}{16}$

(5)

Example. $\sqrt[n]{n!} \leq \left(\frac{n+1}{2}\right)$ for $n = 1, 2, 3, \ldots$.

Proof $n! = 1 \times 2 \times 3 \times \cdots \times n$ and

$$\frac{1+2+3+\cdots+n}{n} = \frac{n(n+1)/2}{n} = \frac{n+1}{2}$$

so $n! \leq \left(\frac{n+1}{2}\right)^n$.

[When $n = 4$, this gives $24 \leq \left(\frac{5}{2}\right)^4 = \frac{625}{16} = 39\frac{1}{16}$.]

Example: If $x_1, x_2, \ldots, x_n$ are positive real numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$ Harmonic mean

Proof Let $y_i = \frac{1}{x_i}$, $i = 1, 2, \ldots, n$ and then apply the AGM with $x_1, \ldots, x_n$ replaced by $y_1, y_2, \ldots, y_n$. $\sqrt[n]{y_1 y_2 \cdots y_n} \leq \left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right)$

Example, $n = 3$: $\sqrt[3]{30} > \frac{3}{\frac{1}{2} + \frac{1}{3} + \frac{1}{5}} = \frac{90}{31}$.

$$\sqrt[n]{\frac{1}{x_1 x_2 x_n}} \leq \frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n}$$

$$n \leq \sqrt[n]{\cdot}\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

Exercises

1. Find real numbers a, b, c with $a+b+c = 3$ and $abc = 10$ or prove that such a, b, c do not exist.

2. Let $n \geq 2$ be an integer. Prove that
$$\sqrt[n]{n} < 1 + \sqrt{\tfrac{2}{n}}.$$
[Hint: Let $a = \sqrt[n]{n} - 1$, so $n = (1+a)^n$ and use the binomial theorem].

3. Let $x_1, \ldots, x_n$ be positive real numbers with $\sum_{i=1}^{n} x_i = 1$ and let $x_{n+1} = 1$.
Prove that $$\sum_{j=1}^{n} \left(\frac{x_j^2}{x_j + x_{j+1}} \right) \geq \frac{1}{2}.$$
[Hint: $x_j^2 = x_j(x_j + x_{j+1}) - x_j x_{j+1}$].

4. Let $n \geq 2$ be an integer. Prove that
$$\frac{1}{2} < \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < 1.$$
[Note that you must prove separately that the middle expression is greater than $\frac{1}{2}$ and less than 1]

5. Determine with proof (not by just looking at the decimal expansions) which is bigger a or b in each of the following cases:
(i) $a = 7^{1/8}$, $b = 8^{1/7}$; (ii) $a = \sqrt{101} - \sqrt{99}$, $b = \frac{1}{20}$,
(iii) $a = \sqrt{2} + \sqrt[3]{3}$, $b = \sqrt[4]{66}$, (iv) $a = \log_2 3$, $b = \log_3 5$.

Some useful algebraic factorizations

(1) For n a positive integer

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$$

The cases $a^2 - b^2 = (a-b)(a+b)$ and

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

are often encountered in school.

To prove (1), just multiply out the right hand side and observe all the cancellations that occur.

(2) For n an odd positive integer

$$a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1})$$

For example, $a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

To prove (2), replace b by $-b$ in (1) above and note that $(-b)^m = (-1)^m b^m$ for each integer m.

(3) $(x+y+z)^3 - x^3 - y^3 - z^3 = 3(x+y)(y+z)(z+x)$.

[Proof: Multiply out both sides]

(4) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2 - xy - yz - zx)$.

[Proof: Multiply out the right-hand-side]

(5) $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$.

[Proof: $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2+2)^2 - (2x)^2$]

(6) $(x+y)^5 - x^5 - y^5 = 5xy(x+y)(x^2 + xy + y^2)$.

(7) $(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2 + xy + y^2)^2$.

(8) $x^4 + x^2y^2 + y^4 = (x^2 - xy + y^2)(x^2 + xy + y^2)$.

[Proof: $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2 = (x^2+y^2)^2 - (xy)^2$]

(9) $x^n - 1 = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1})$ where $\omega = \cos\left(\frac{2\pi}{2n}\right) + i\sin\left(\frac{2\pi}{2n}\right)$ and $i = \sqrt{-1}$.

(10) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$