UCD School of Mathematics and Statistics Summer Intership 2019

Orbits of a Spinning Test Particle about a Kerr Black Hole

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July 31, 2019

1 Introduction

We set out to produce Mathematica notebooks to calculate generic orbits of forced test bodies around a Kerr black hole. For the sake of this code we have used the linear order MPD equations as our force, to describe the deviation of a spinning test body from the Kerr geodesics. However, the form of our code means including different forces is a straightforward procedure. What follows is a brief introduction to geodesics, the MPD equations, the oscullating geodesic and Null-Tetrad formulations, and a short note on the choice of initial conditions used. In particular, this report draws together the works of Warburton et al. [1] and Gair et al. [2].

2 Spacetime

2.1 The Metric in 3 Dimensions

In Euclidean space, we intuitively know the shortest path between two points to be a straight line. However, this notion is no longer useful once we attempt to generalize the idea of shortest distances to curved spaces (or indeed as we shall see, space**times**).
We therefore quickly build up our notation to describe distances between points in general spaces.
We define the infinitesimal line element along our surface, by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.1}$$

where $g_{\mu\nu}$ known as the *metric tensor*, which encodes the curvature of our space. For Euclidean space the metric is simply represented by the identity matrix,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

so we have

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{2.3}$$

So we recover the familiar pythagorean rule for straight line paths.

As a brief example of how this applies to curved surfaces, on the unit sphere we have the line element (in spherical polar coordinates)

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \tag{2.4}$$

.

The arclength between two points can now be obtained by integrating the line element

$$S = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda \tag{2.5}$$

Where we have parametrised our path through space by some parameter λ. Solving the Euler-Lagrange equations of this distance functional yields our path of shortest distance, known as a **geodesic**. In Euclidean space, this path is the familiar straight line; on the sphere, a great circle. However in general curved spacetimes as we shall now see, the paths are no longer straightforward.

2.2 Geodesic motion in Curved Spacetime

In relativistic kinematics, we no longer have a notion of absolute time for all observers. Thus, Einstein found it necessary to place the time measured by an observer on equal footing with the spatial coordinates. Therefore, our metric now encodes not only the curvature of *space*, but also of time! So our definition Eqn. (2.1) now has the indices ranging from 0 to 3, where x^0 is the coordinate-time component t. To find our Spacetime geodesics then, we may vary the action

$$S = \int g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda \tag{2.6}$$

From which we have

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \tag{2.7}$$

These equations are known as the *geodesic equations* and describe the motion of an unforced test particle in the spacetime determined by the metric (Note: $\Gamma^\alpha_{\mu\nu}$ is known as the *Connection* and allows the generalisation of a derivative to curved spacetimes, a definition of which can be found on page 93 of [3]). The metric itself is determined by **Einstein's Equation**

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.8}$$

Where, loosely speaking, the curvature (and indeed the metric) of the spacetime is related to its energy and momentum content (encapsulated in the stress-energy tensor, $T_{\mu\nu}$). Two exact solutions of this equation - the Schwarzschild and Kerr solutions - describe the spacetime about stationary, respectively non-rotating and rotating black holes, and it is the geodesics of these spacetimes to which we now turn.

3 Geodesics

3.1 The Schwarzschild Solution

The first important solution to Einstein's Equation (2.8) is the spacetime described by a stationary, non-rotating black hole - the Schwarzschild Solution, with metric

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \tag{3.1}$$

where $f = 1 - \frac{2M}{r}$, working in geometrised units where $G = c = 1$. The coordinates here are the familiar spherical polar coordinates $\{t, r, \theta, \phi\}$.

As the metric has no t- or ϕ -dependence two Killing vectors corresponding to conserved quantities are admitted

$$\xi^{(t)} = \partial_t,\ \xi^{(\phi)} = \partial_\phi \tag{3.2}$$

As a consequence of Noether's Theorem we therefore define the conserved quantities.

$$E := -\xi^{(t)}_\mu \frac{dx^\mu}{d\lambda} = f\frac{dt}{d\lambda} \tag{3.3}$$

and

$$L_z := \xi^{(\phi)}_\mu \frac{dx^\mu}{d\lambda} = r^2 \frac{d\phi}{d\lambda} \tag{3.4}$$

As in [3], these are the energy and angular momentum per unit mass, respectively.

The geodesic equations (2.7) for the Schwarzschild spacetime, after substituting in our conserved E and L_z, are given by

$$\begin{cases} (\frac{dr}{d\lambda})^2 - E^2 + (1 - \frac{2M}{r})(\frac{L_z^2}{r^2} + 1) = 0 \\ \frac{d}{d\lambda}(r^2 \frac{d\theta}{d\lambda}) = \frac{L_z^2 \sin\theta\cos\theta}{r^2} \end{cases} \tag{3.5}$$

, Where λ is an affine parameter related to the proper time τ of the orbiting body by

$$\lambda = a\tau + b \tag{3.6}$$

for constants a, b. However if we set our initial conditions for θ to $\theta(0) = \frac{\pi}{2}, \dot{\theta}(0) = 0$, we note that $\ddot{\theta} = 0$ for all time i.e. our motion is confined to a plane through the centre of the black hole. This is exactly commensurate with our intuition, due to the spherical symmetry of the spacetime.

Thus, we need only solve the radial equation:

$$\left(\frac{dr}{d\lambda}\right)^2 - E^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{L_z^2}{r^2} + 1\right) = 0 \tag{3.7}$$

For the purposes of solving this equation, we chose to parametrise our solution space not by the constants of the motion $\{E, L_z\}$, but by the orbital eccentricity e and semi-latus rectum p (there is a one-to-one correspondence between the parameters $\{E, L_z\}$ and $\{p, e\}$, formulae for which are given in [1]).

3.2 The Kerr Solution

The Schwarzschild solution, while important, does not give a very astrophysically realistic example of a black hole spacetime, as we would not expect a black hole to be perfectly non-rotating. Therefore, we now examine geodesics in the spacetime of a stationary, axially symmetric, rotating black hole known as *Kerr Spacetime*, described by the metric

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{2Mar\sin^2\theta}{\Sigma}(\mathrm{d}t\mathrm{d}\phi + \mathrm{d}\phi\mathrm{d}t) \\ &+ \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\Sigma}[\varpi^4 - a^2\Delta\sin^2\theta]\mathrm{d}\phi^2 \end{aligned} \tag{3.8}$$

in Boyer-Lindquist coordinates (cf. pg. 262 of [3]), where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$ and $\varpi = \sqrt{r^2 + a^2}$. The parameter a is the angular momentum per unit mass of the black hole.

As in the Schwarzschild case, the metric admits the killing vectors given in (3.2), and thus has a conserved energy and angular momentum associated with each geodesic. However this metric also admits a third conserved quantity known as the *Carter Constant*:

$$Q = a^2(1 - E^2)\sin^2\iota + L_z^2\tan^2\iota \tag{3.9}$$

where ι is the angle of inclination of the orbit above the equatorial plane.

Armed with the conserved energy, angular momentum and Carter constant, we can simplify the geodesic equations for Kerr spacetime to the equations given by [4]:

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 &= [E(r^2 + a^2) - aL_z]^2 - \Delta[r^2 + (L_z - aE)^2 + Q] \\ &\equiv V_r(r), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \left(\frac{d\theta}{d\lambda}\right)^2 &= Q - \cot^2\theta L_z^2 - a^2\cos^2\theta(1 - E^2) \\ &\equiv V_\theta(\theta), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \frac{d\phi}{d\lambda} &= \csc^2\theta L_z + aE\left(\frac{r^2 + a^2}{\Delta} - 1\right) - \frac{a^2 L_z}{\Delta} \\ &\equiv V_\phi(r, \theta), \end{aligned} \tag{3.12}$$

$$\frac{dt}{d\lambda} = E\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right] + aL_z\left(1 - \frac{r^2+a^2}{\Delta}\right)$$
$$\equiv V_t(r,\theta), \tag{3.13}$$

Where λ is known as *Mino time*, related to the proper time of the orbiting body by [4]:

$$d\lambda = \frac{1}{\Sigma} d\tau \tag{3.14}$$

This decouples the radial and polar equations, so that V_r and V_θ are functions only of r and θ, respectively. To solve these equations numerically for stable bound timelike orbits we implemented code to convert the parameters $\{e, p, \theta_{min}\}$ to the set $\{E, L_z, Q\}$. This work makes use of the Black Hole Perturbation Toolkit [5].

4 The Mathisson-Papapetrou-Dixon Equations

In the case of a zero-spin body orbiting a black hole, it is typical to model the body as a point particle. However, the case of a spinning particle is more complicated, because the internal structure of the particle must be taken into account. In 1951 Papapetrou did this by modelling test particles as multipoles [6], where a point particle is a modelled as a monopole, and a spinning particle as a dipole. Higher order terms can be taken, but their importance drops off rapidly. From this starting point the Mathisson-Papapetrou-Dixon (MPD) equations for a spinning test body can be derived. These are shown below, as described in [1, 7]:

$$\frac{Dp^\alpha}{d\tau} = u^\beta \nabla_\beta p^\alpha = -\frac{1}{2} R^\alpha{}_{\nu\lambda\sigma} u^\nu S^{\lambda\sigma} \tag{4.1}$$

$$\frac{DS^{\mu\nu}}{d\tau} = u^\beta \nabla_\beta S^{\mu\nu} = p^\mu u^\nu - p^\nu u^\mu \tag{4.2}$$

Where $S^{\mu\nu}$ is the spin tensor related to the the spin vector as:

$$S^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} u_\alpha S_\beta \tag{4.3}$$

$$\epsilon_{\mu\nu\alpha\beta} = \sqrt{-g}[\alpha\beta\gamma\delta] \tag{4.4}$$

where $[\alpha\beta\gamma\delta]$ is the totally anti-symmetric symbol and g is the determinant of the metric.

$$g = \underbrace{-\Sigma^2\sin^2\theta}_{\text{Kerr Case}} = \underbrace{-r^4\sin^2\theta}_{\text{Schwarzschild Case}} \tag{4.5}$$

Note that the right hand side of Eqn (4.2) does not vanish, as in general, the momentum of a spinning particle is not directly proportional to it's velocity, but is given by:

$$p^\alpha = \mu u^\alpha - u_\gamma \frac{DS^{\alpha\gamma}}{d\tau} \tag{4.6}$$

However we see that if we linearize the system in S (i.e. neglect terms $\mathcal{O}(S^2)$ and higher), our equations simplify significantly. Eqn. (4.6) becomes $p^\alpha = \mu u^\alpha$, which causes Eqn. (4.2) to become:

$$\frac{DS^{\mu\nu}}{d\tau} = u^\beta \nabla_\beta S^{\mu\nu} = 0 \tag{4.7}$$

So we see that the spin tensor is parallel transported, at least at first order. Furthermore, we recognise that the left hand side of Eqn. (4.1) becomes the general equation for a force,

$$\mu u^\beta \nabla_\beta u^\alpha = F^\alpha \tag{4.8}$$

So we see that the effect of spin manifests itself as a force, given by:

$$F^\alpha_{\text{Spin-Curvature}} = -\frac{1}{2} R^\alpha{}_{\nu\lambda\sigma} u^\nu S^{\lambda\sigma} \tag{4.9}$$

By combining Eqns. (4.3), (4.7), we can see that:

$$\frac{dS^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu} S^\mu u^\nu \tag{4.10}$$

Or in Mino time (λ), for the Kerr case:

$$\frac{dS^\alpha}{d\lambda} = -\Gamma^\alpha_{\mu\nu} S^\mu U^\nu, \quad U^\nu = \frac{dx^\nu}{d\lambda} \tag{4.11}$$

As a consistency check for later on we note that :

$$S^2 = S^\alpha S_\alpha = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta} \tag{4.12}$$

should be constant. In particular, when our test body is itself a Kerr black hole, we expect to see $S = s\mu^2$, where $0 < |s| < 1$. Furthermore, the conserved quantities admitted by the Killing vectors must be altered to account for the spin.

$$\mathcal{C}^S = p_\alpha \xi^\alpha - \frac{1}{2} S^{\alpha\beta} \nabla_\beta \xi_\alpha \tag{4.13}$$

This equation will allow us to evaluate the energy and angular momentum of the system.

Tracing the worldline in the case of the non-spinning body is simple, as we can model the body as a point particle. However, in the spinning case, we must take into account the internal structure of the particle. So we pick a point on our particle to trace it's worldline. The obvious choice of point is the center of mass of the body, but due to relativistic effects, observers in different reference frames will disagree on the position of the center of mass. We used a spin supplementary condition (SSC) [8] to specify this point. In particular we have used the Pirani Condition, $u_\mu S^{\mu\nu}$.

5 The Method of Oscullating Geodesics

Since the spin manifests itself as a force acting on the test-particle, the particle is forced off of a geodesic path, so it's worldline is no longer determined by the geodesic equation. To model it's new path we note that at each point along it's worldline, the particle is moving tangentially to some geodesic. By evolving the orbital parameters to mimic how the particle moves from one instantaneous geodesic to another, we can find the path followed by the particle.
Given a worldline $z^\alpha(\tau)$, parameterised by proper time, τ, then, as described by Pound and Poisson in [9]:

$$\frac{\partial z_G^\alpha}{\partial I^A}\dot{I}^A = 0 \tag{5.1}$$

$$\frac{\partial \dot{z}_G^\alpha}{\partial I^A}\dot{I}^A = F^\alpha \tag{5.2}$$

where $z_G^\alpha(I^A,\tau)$ is a geodesic with the set of orbital parameters I^A.

6 The Schwarzschild Case

In this section we rely heavily on the work done by Warburton, Osburn, and Evans in [1]. We shall denote equatorial geodesics as $z_G'^\alpha(I^A,\tau) = \{t(\tau), r(\tau), \pi/2, \varphi'(\tau)\}$, where the prime is dropped on r and t, as they are invariant under rotation. The square on $dr/d\tau$ from the geodesic equations of motion (3.5) introduces awkward turning points, so instead we choose to define r in terms of the relativistic anomaly, χ:

$$r(\chi) = \frac{pM}{1+e\cos(\chi-\chi_0)} \tag{6.1}$$

Where we will later denote $\nu = \chi - \chi_0$. We can find the relationship between τ and χ can be found by taking $d\tau/d\chi = \frac{dr/d\chi}{dr/d\tau}$:

$$\frac{d\tau}{d\chi} = \frac{Mp^{3/2}}{(1+e\cos\nu)^2}\sqrt{\frac{p-3-e^2}{p-6-2e\cos\nu}} \tag{6.2}$$

From there we can derive the derivatives of the other co-ordinates:

$$\frac{dt}{d\chi} = \frac{r^2}{M(p-2-2e\cos\nu)}\sqrt{\frac{(p-2)^2-4e^2}{p-6-2e\cos\nu}} \tag{6.3}$$

$$\frac{d\varphi'}{d\chi} = \sqrt{\frac{p}{p-6-2e\cos\nu}} \tag{6.4}$$

t will be evaluated dynamically , but it is more useful to rewrite φ' in terms of the incomplete elliptic integral of the first kind, and an initial condition Φ:

$$\varphi'(\chi) = \Phi + 2\sqrt{\frac{p}{p-6-2e}}F\left(\frac{\nu}{2}\middle|\frac{-4e}{p-6-2e}\right) \tag{6.5}$$

Now, we can convert from $z_g'^\alpha$ to z_g^α, via a rotation through the Euler angles Ω and ι. the radial and temporal co-ordinates remain unchanged, but our azimuthal and polar angles become:

$$\varphi(\chi) = \Omega + \int_0^{\varphi'}(\sec\iota\cos^2 u + \cos\iota\sin^2 u)^{-1}du \tag{6.6}$$

$$\theta = \cos^{-1}(\sin\iota\sin\varphi') \tag{6.7}$$

Finally the r and t components of u_G^α remain unchanged, but the φ and θ components are rotated to become:

$$u_G^\theta = -\frac{(1+e\cos\nu)^2\sin\iota\cos\varphi'}{pM\sqrt{(p-3-e^2)(1-\sin^2\iota\sin^2\varphi')}} \tag{6.8}$$

$$u_G^\varphi = \frac{(1+e\cos\nu)^2(p-3-e^2)^{-1/2}}{pM(\sec\iota\cos^2\varphi'+\cos\iota\sin^2\varphi')} \tag{6.9}$$

to fully describe any of our instantaneous geodesics we need the parameters $I^A = \{e, p, \chi_0, \iota, \Omega, \Phi\}$. To evolve these with χ we call on Eqns. (5.1), (5.2). By solving this system of equations we find expressions for $\frac{\partial e}{\partial\chi}, \frac{\partial p}{\partial\chi}, \frac{\partial\chi_0}{\partial\chi}, \frac{\partial\iota}{\partial\chi}, \frac{\partial\Omega}{\partial\chi}, \frac{\partial\Phi}{\partial\chi}$. Once we have found all these expressions it is simple to numerically evaluate the path followed by a particle.

According to Eqn. (4.13), the total energy and z- component of the angular momentum of the system in the Schwarzschild case are given by the expressions:

$$E^S = E^G + \frac{M}{\mu}\sin\theta(u^\theta S^\phi - u^\phi S^\theta) \tag{6.10}$$

$$E^G = \sqrt{\frac{(p-2)^2 - 4e^2}{p(p-3-e^2)}} \tag{6.11}$$

$$L_z^S = L_z^G + \frac{1}{\mu}\Big(fr(S^t u^\theta - S^\theta u^t)\sin\theta + (S^r u^t - S^t u^r)\cos\theta\Big) \tag{6.12}$$

$$L_z^G = r^2 u^\varphi \sin^2\theta \tag{6.13}$$

We shall also take advantage of our SSC and Eqn. (4.3) to eliminate a degree of freedom in choosing initial conditions for the spin:

$$u_\mu S^\mu = 0 \Rightarrow$$
$$S^t = \frac{S^r u^r + f\, r^2(S^\theta u^\theta + S^\varphi u^\varphi \sin^2\theta)}{f^2 u^t} \tag{6.14}$$

Similarly, we check $S^\alpha S_\alpha$ to ensure that 0<|s|<1, and that our system is physically realistic.

6.1 The Spin-Aligned System

The simplest case we modelled was a system where the spin of the test body lined up parallel to it's orbital angular momentum. In this case ι and Ω are constant, and we dynamically evaluate φ' using Eqn. (6.4). This corresponds to a planar orbit. We can arbitrarily choose $\iota = 0$, which gives $\theta = \pi/2$, and $\Omega = 0$. For the spin to be aligned, the spin must point entirely in the θ direction, i.e. $S^\alpha = \{0, 0, S^\theta, 0\}$, and from Eqn. (4.10), we show that:

$$S^\theta = \frac{s\mu^2}{r} \tag{6.15}$$

where s is a measure of the spin of the test body, such that $0 < |s| < 1$, and μ is the mass of the test body.

The forces in this case are:

$$F^t_{\text{Spin-Curvature}} = -\frac{3s\mu^2 M u^r u^\varphi}{r^2 f} \tag{6.16}$$

$$F^r_{\text{Spin-Curvature}} = -\frac{3s\mu^2 M f u^t u^\varphi}{r^2} \tag{6.17}$$

Initial values for s, e, p and χ_0 were taken, corresponding to bound geodesic orbits. Expressions for $\frac{\partial e}{\partial \chi}, \frac{\partial p}{\partial \chi}, \frac{\partial \chi_0}{\partial \chi}$ were coded into Mathematica, and evaluated numerically using NDSolve. The following are some plots created once we programmed this model into Mathematica.

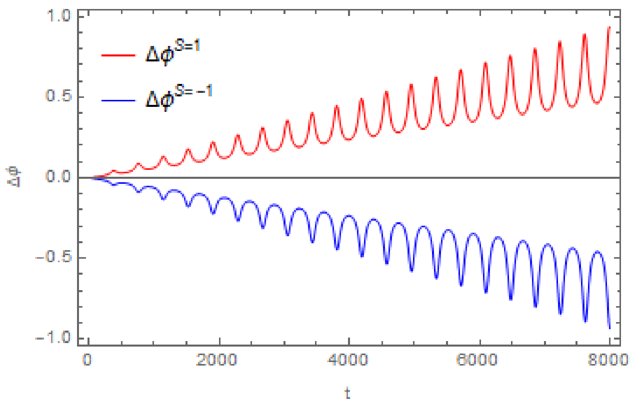

Figure 1: Change in phase of spinning and non-spinning particle. In each case the phase difference is the $\varphi^{S\neq 0} - \varphi^{S=0}$. The inital conditions are $e_0 = 0.4$, $p_0 = 10$, and the mass ratio is $\mu/m = 5 \times 10^{-3}$.

We can see from Fig. [1] that a positive spin (i.e. Spin-aligned) leads to an increase in the rate of precession, while a negative spin (i.e. Spin-anti-aligned) leads to a decrease in the rate of precession. This can be seen clearly in the following diagram as well:

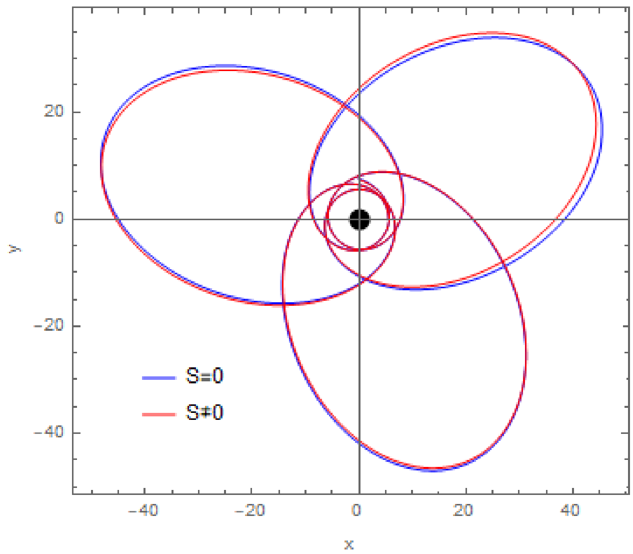

Figure 2: Path of a maximally spinning particle (s=1) around a Schwarzschild black Hole, where the orbital angular momentum and spin are parallel. Note that after 2-3 orbits the dephasing is appreciable. The inital conditions are $e_0 = 0.8$, $p_0 = 10$, and the mass ratio is $\mu/m = 5 \times 10^{-3}$.

We also note that energy and angular momentum are conserved to six significant figures, at which scale the numerical error of our numerical solver is non-negligible.

6.2 Spin-Misaligned case

The Spin-Misaligned case is a simple extension of the previous case. By dropping our assumptions that ι and Ω are constant, and by using Eqn. (6.5) to find φ', and evolving Φ with χ. Our Spin vector is no longer confined to the θ direction, and also precesses with χ. We will now have four forces:

$$F^t_{Spin-Curvature} = \frac{3Mu^r \sin\theta(S^\varphi u^\theta - S^\theta u^\varphi)}{rf} \quad (6.18)$$

$$F^r_{Spin-Curvature} = \frac{3Mfu^t \sin\theta(S^\varphi u^\theta - S^\theta u^\varphi)}{r} \quad (6.19)$$

$$F^\theta_{Spin-Curvature} = \frac{3Mu^\varphi \sin\theta(S^t u^r - S^r u^t)}{r^3} \quad (6.20)$$

$$F^\varphi_{Spin-Curvature} = -\frac{3Mu^\theta \sin\theta(S^t u^r - S^r u^t)}{r^3 \sin\theta} \quad (6.21)$$

the spin will change as described by Eqn. (4.10), which simplifies in the Schwarzschild case to:

$$\frac{dS^t}{d\chi} = -\frac{d\tau}{d\chi}\frac{M}{f\,r^2}(S^t u^r + S^r u^t) \quad (6.22)$$

$$\frac{dS^r}{d\chi} = \frac{d\tau}{d\chi}\left(\frac{M}{f\,r^2}(S^r u^r f^2 S^t u^t) + f\,r(S^\theta u^\theta + S^\varphi u^\varphi \sin^2\theta)\right) \quad (6.23)$$

$$\frac{dS^\theta}{d\chi} = \frac{d\tau}{d\chi}\left(S^\varphi u^\varphi \cos\theta\sin\theta - \frac{S^\theta u^r + S^r u^\theta}{r}\right) \quad (6.24)$$

$$\frac{dS^\varphi}{d\chi} = -\frac{d\tau}{d\chi}\left(\cot\theta(S^\varphi u^\theta + S^\theta u^\varphi) + \frac{S^r u^\varphi + S^\varphi u^r}{r}\right) \quad (6.25)$$

The following are plots generated in Mathematica to show the interesting features of this system.

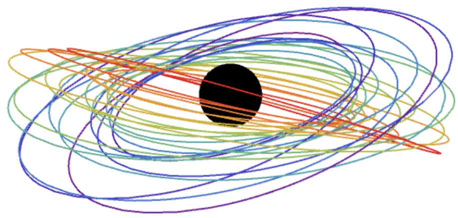

Figure 3: Path of a spinning particle around a Schwarzschild Black hole, where the orbital angular momentum and spin are not parallel. The departure from the initial plane is significant. The inital conditions are $e_0 = 0.3$, $p_0 = 10$, $\iota_0 = \pi/4$, and the mass ratio is $\mu/m = 10^{-2}$. This is probably too large a mass ratio for the model, but for the sake of demonstration it illustrates the precession of the plane very well. The colour represents evolution in the relativistic anomaly, χ, starting in dark blue and working up to red.

Again, we see that the energy and z-component of angular momentum are conserved to 6 Significant figures, the accuracy of our numerical solver.

7 Kinnersley Null-Tetrad Formulation

Our programme for the spinning test body orbiting a Kerr black hole implements the work of Gair et al. in [2].
First of all, we will parameterize our motion in the r and θ co-ordinates by two anomalies, ψ_r and ψ_θ. In this case we will have:

$$z = \cos\theta = \sqrt{z_-}\cos\psi_\theta \quad (7.1)$$

Where z_- is the smaller root of the polar effective potential V_θ . Similarly we will define:

$$r = \frac{pM}{1 + e\cos\psi_r} \quad (7.2)$$

Where e and p are described by:

$$e = \frac{r_1 - r_2}{r_1 + r_2}, \quad p = \frac{2r_1 r_2}{r_1 + r_2} \quad (7.3)$$

And r_1 and r_2 are the two greatest roots of the radial potential such that $r_2 < r < r_1$.

We also use the equations derived by Gair et al. in [2]:

$$\frac{dE}{d\lambda} = \frac{u_r a_n \Delta}{u_n} - \frac{\Delta \mathcal{A}_{III}}{2u_n} - a\sin\theta \mathcal{A}_{II} \quad (7.4)$$

$$\frac{dL_z}{d\lambda} = \frac{a\sin^2\theta u_r a_n \Delta}{u_n} - \frac{a\sin^2\theta \Delta \mathcal{A}_{III}}{2u_n} - \varpi^2 \sin\theta \mathcal{A}_{II} \quad (7.5)$$

$$\frac{dK}{d\lambda} = 2\Sigma^2 \mathcal{A}_{III} \quad (7.6)$$

$$K = Q + (L_z - aE)^2 \quad (7.7)$$

Where we are working in the Null-Tetrad formulation so that:

$$u_l = u_r - \frac{F}{\Delta} \quad (7.8)$$

$$u_n = -\frac{F}{2\Sigma} - \frac{\Delta}{2\Sigma} u_r \quad (7.9)$$

$$R_u = \frac{u_m + u_m^*}{\sqrt{2}} = \frac{r}{\Sigma} u_\theta + \frac{a\mathcal{H}\cos\theta}{\Sigma\sin\theta} \quad (7.10)$$

$$I_u = \frac{i(u_m - u_m^*)}{\sqrt{2}} = \frac{a\cos\theta}{\Sigma} u_\theta - \frac{r\mathcal{H}}{\Sigma\sin\theta}$$

(7.11)

$$a_n = \frac{\varpi^2}{2\Sigma} a_t - \frac{\Delta}{2\Sigma} a_r + \frac{a}{2\Sigma} a_\phi \quad (7.12)$$

$$a_l == \frac{\varpi^2}{\Delta} a_t + a_r + \frac{a}{\Delta} a_\phi \quad (7.13)$$

$$a_m = \frac{1}{\sqrt{2}(r - ia\cos\theta)} \left(ia\sin\theta a_t + a_\theta + \frac{i}{\sin\theta} a_\phi \right) \quad (7.14)$$

$$R_a = \frac{a_m + a_m^*}{\sqrt{2}} \quad (7.15)$$

$$= \frac{a^2 \sin\theta\cos\theta}{\Sigma} a_t + \frac{r}{\Sigma} a_\theta + \frac{a\cot\theta}{\Sigma} a_\phi \quad (7.16)$$

$$I_a = \frac{i(a_m + a_m^*)}{\sqrt{2}} \quad (7.17)$$

$$= -\frac{ar\sin\theta}{\Sigma} a_t - \frac{a\cos\theta}{\Sigma} a_\theta - \frac{r}{\Sigma\sin\theta} a_\phi \quad (7.18)$$

We will also have our Null-Tetrad accelerations in terms of the Boyer-Lindquist co-ordinates:

$$\mathcal{A}_I = a_\theta \quad (7.19)$$

$$\mathcal{A}_{II} = -a\sin\theta a_t - \frac{1}{\sin\theta} a_\phi \quad (7.20)$$

$$\mathcal{A}_{III} = \frac{a(L_z - aE\sin^2\theta)}{\Sigma} a_t + \frac{u_\theta}{\Sigma} a_\theta + \frac{(L_z - aE\sin^2\theta)}{\Sigma\sin^2\theta} a_\phi \quad (7.21)$$

Finally, we will need to know how or anomalies ψ_r and ψ_θ change with Mino time.

$$\begin{aligned}\frac{d\psi_r}{d\lambda} =&\mathcal{P} + \frac{\mathcal{C}\mathcal{A}_{III}\sin\psi_r}{2(1+e\cos\psi_r)u_n} + \frac{\mathcal{D}\Sigma\mathcal{A}_{III}\mathcal{P}}{2(1+e\cos\psi_r)^2 u_n} - \frac{a\mathcal{E}\sin\theta\sin\psi_r\mathcal{A}_{II}}{1+e\cos\psi_r} + \\ &\frac{\mathcal{P}a_n}{u_n(1+e\cos\psi_r)^2}\left[(1-e^2)(1-\cos\psi_r)\frac{\Sigma_1 F_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r)\frac{\Sigma_2 F_2}{\kappa_2}\right]\end{aligned} \tag{7.22}$$

$$\frac{d\psi_\theta}{d\lambda} = \sqrt{\beta(z_+ - z)}\left[1 + \frac{(1-z_-)\Sigma\mathcal{A}_I\cos\psi_\theta}{\beta\sqrt{z_-}(z_+ - z_-)\sin\theta}\right] + \frac{\cos\psi_\theta\sin\psi_\theta\mathcal{H}a\Delta(\mathcal{A}_{III} - 2u_r a_n)}{2(z_+ - z_-)\beta u_n} + \frac{\cos\psi_\theta\sin\psi_\theta\mathcal{G}\mathcal{A}_{II}}{\beta(z_+ - z_-)} \tag{7.23}$$

Note the following shorthands have been used:

$$\begin{aligned}\mathcal{Q}_1 &= -2aL_z rr_1 - a^4E(r+r_1) + a^3L_z(r+r_1) - a^2E(r^3 + r^2r_1 + r_1^3 + rr_1(r_1 - 2)) \\ &-Err_1(rr_1(r+r_1) - 2(r^2 + rr_1 + r_1^2)) - a^2(2a^2E - 2Err_1 + aL_z(r + r_1 - 2))\cos^2\theta\end{aligned} \tag{7.24}$$

$$\begin{aligned}\mathcal{Q}_2 &= -2aL_z rr_2 - a^4E(r+r_2) + a^3L_z(r+r_2) - a^2E(r^3 + r^2r_2 + r_2^3 + rr_2(r_2 - 2)) \\ &-Err_2(rr_2(r+r_2) - 2(r^2 + rr_2 + r_2^2)) - a^2(2a^2E - 2Err_2 + aL_z(r + r_2 - 2))\cos^2\theta\end{aligned} \tag{7.25}$$

In the following equations a subscript of 1 or 2 implies that the expression is evaluated at r_1 or r_2, unless already defined uotherwise.

$$\mathcal{H} = L_z - aE\sin^2\theta \tag{7.26}$$

$$F = \varpi^2 E - aL_z \tag{7.27}$$

$$\mathcal{P} = \frac{p\sqrt{\mathcal{J}}}{1-e^2} \tag{7.28}$$

$$\begin{aligned}\mathcal{J} &= (1-E^2)(1-e^2) + 2\left(1 - E^2 - \frac{1-e^2}{p}\right)(1+e\cos\psi_r) + \\ &\left[(1-E^2)\frac{3+e^2}{1-e^2} - \frac{4}{p} + [a^2(1-E^2) + L_z^2 + Q]\frac{1-e^2}{p^2}\right](1+e\cos\psi_r)^2\end{aligned} \tag{7.29}$$

$$\mathcal{C} = \frac{\mathcal{Q}_1(1-e)}{\kappa_1} - \frac{\mathcal{Q}_2(1+e)}{\kappa_2} \tag{7.30}$$

$$\mathcal{D} = (1-e)^2(1-\cos\psi - r)\frac{\Delta_1}{\kappa_1} + (1+e)^2(1+\cos\psi_r)\frac{\Delta_2}{\kappa_2} \tag{7.31}$$

$$\mathcal{E} = \frac{F_1(1-e)(r+r_1}{\kappa_1} - \frac{F_2(1+e)(r+r_2)}{\kappa_2} \tag{7.32}$$

$$\kappa = 4EFr - 2r\Delta - 2(r-M)(r^2+K) \tag{7.33}$$

We again use our SSC to remove a degree of freedom from the initial conditions for spin, and we use the contraction $S^\alpha S_\alpha$ to check whether the magnitude of the spin is realistic. We define our anomalies so that $\psi_r(0) = \psi_\theta(0) = 0$. We generate initial values for E, L_z and Q, from values of e, p and ι corresponding to a bound geodesic orbit. Then by evolving E, L_z, K, ψ_r, ψ_θ and S^α, using Eqns. (4.10), (7.4), (7.5), (7.6) we can see how the orbit changes over time. By using the fact that e and p are functions of r_1 and r_2, which in turn are functions of E, L_z and K, as is z_m, we have all the information we need to plot the orbits of a spinning test body in orbit around a Kerr Black hole. We can see that the spinning worldline deviates quite quickly from the geodesic case:

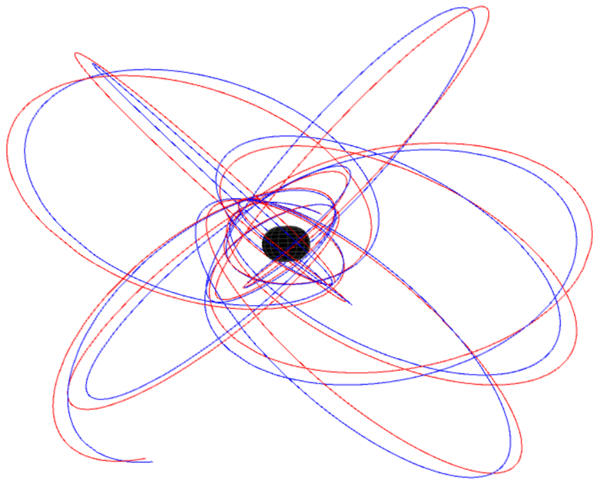

Figure 4: Worldlines of spinning and non-spinning particles orbiting a Kerr Black hole. The blue line represents the geodesic case (S=0), while the red line indicates the forced case (S $\neq$ 0). In this figure λ runs from 80 to 100. $e_0 = 0.7, p_0 = 8, \theta_0 = \pi/4$. The mass ratio $\mu/M = 0.01$

It is clear to see that the path of the spinning body follows the non-spinning path closely, but deviates slighlty. this deviation accumulates over time. In the Kerr system, the adjusted energy and angular momentum are given by:

$$E^S = E^G + \frac{1}{2\mu}\partial_\beta g_{t\alpha} S^{\alpha\beta} \tag{7.34}$$

$$L_z^S = L_z^G - \frac{1}{2\mu}\partial_\beta g_{\phi\alpha} S^{\alpha\beta} \tag{7.35}$$

We note again that the energy and z-component of the angular momentum are conserved to six significant figures (i.e. the accuracy of our solver).

8 Conclusions

The code we developed successfully computes generic orbits of spinning test bodies about Schwarzschild and Kerr black holes, governed by the linearized MPD equations, and gives appropriately conserved energies and angular momenta for such orbits. Upon optimization, this code could provide efficient calculation of the orbital motion of generically forced bodies.

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Appendix

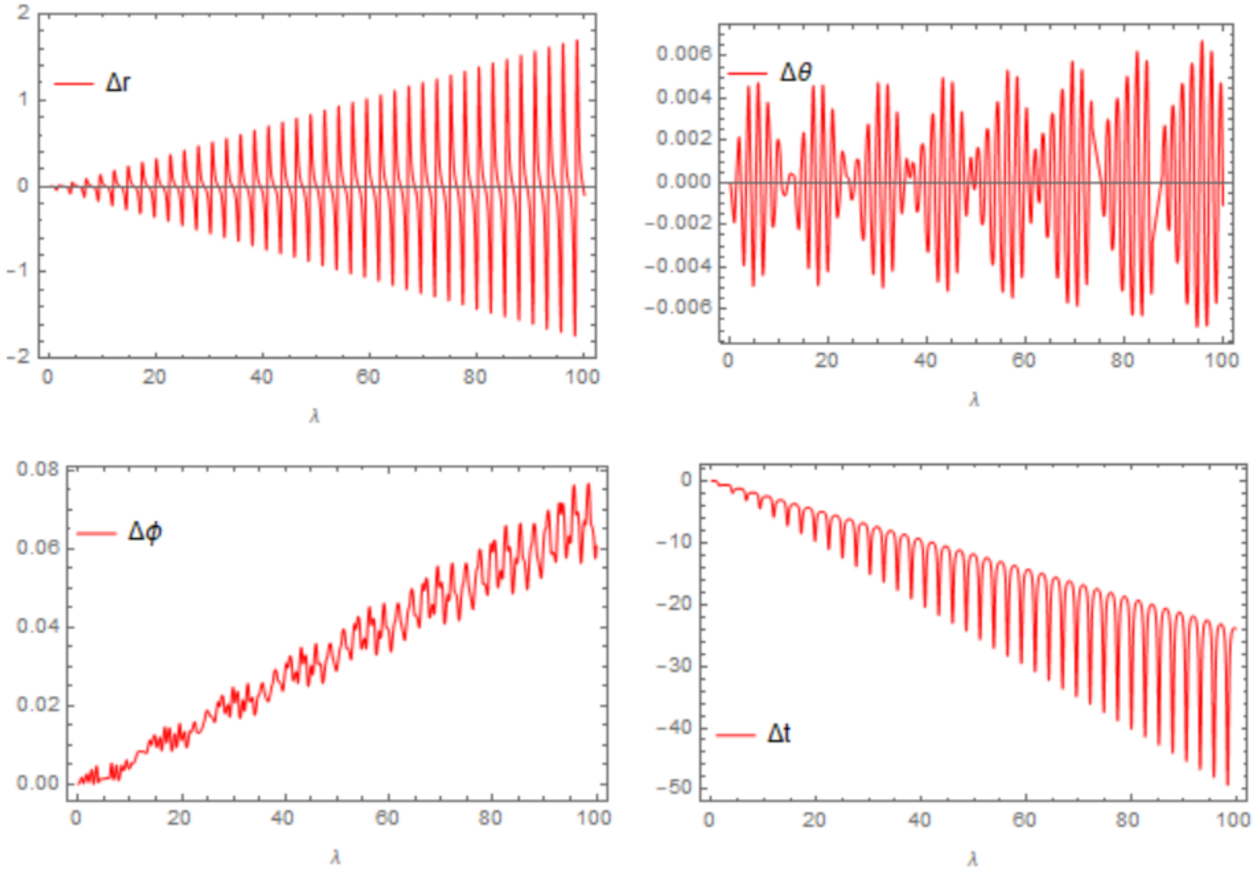

Figure 5: Phase difference in the co-ordinates of a body orbiting a Kerr black hole.
In each case $\Delta x^\mu = x^\mu_{S\neq 0} - x^\mu_{S=0}$. $e_0 = 0.7, p_0 = 8, \theta_0 = \pi/4$. The mass ratio $\mu/M = 0.01$