QUANTUM KNOT INVARIANTS AND MODULARITY

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Quantum knot invariants have their origin in the seminal work of two Fields medalists, Vaughan Jones in 1984 on von Neumann algebras and Edward Witten in 1988 on topological quantum field theory. The equivalence between vacuum expectation values of Wilson loops (in Chern-Simons gauge theory) and knot polynomial invariants (for example, the Jones polynomial, HOMFLY polynomial and Kauffman polynomial) is a major source of inspiration for the development of new knot invariants via quantum groups. A modular form is an analytic object with intrinsic symmetric properties. They have enjoyed long and fruitful interactions with many areas in mathematics such as number theory, algebraic geometry, combinatorics and physics. Their Fourier coefficients contain a wealth of information, for example, about the number of points over the finite field of prime order of elliptic curves, K3 surfaces and Calabi-Yau threefolds. Most importantly, they were the key players in Wiles' spectacular proof in 1994 of Fermat's Last Theorem.

Over the past two decades, there have been many intriguing connections made between quantum invariants of knots and modular forms. For example, quantum invariants of 3-manifolds are typically functions that are defined only at roots of unity and one can ask whether they extend to holomorphic functions on the complex disk with interesting arithmetic properties. A first result in this direction was found in 1999 by Lawrence and Zagier who showed that Ramanujan's 5th order mock theta functions coincide asymptotically with Witten-Reshetikhin-Turaev (WRT) invariants of Poincaré homology spheres. More recently, Zagier has found experimental evidence of modularity properties of a new type for the Kashaev invariants of knots. This "modularity conjecture" implies one of the major outstanding open problems in quantum topology, namely the Volume conjecture. This latter conjecture relates the value at $\zeta_N = e^{2\pi i/N}$ of the Nth colored Jones polynomial (or, equivalently, the Nth Kashaev invariant) of a knot to its hyperbolic volume and, if true, would give striking relations between hyperbolic geometry, quantum topology and modular forms.

The goal of this project will be to study other instances of these intriguing interactions, for example, the stability of the coefficients of the colored Jones polynomial, the explicit construction of new families of q-hypergeometric series arising from the colored Jones polynomial and number theoretic properties of closely related q-hypergeometric series with a view towards proving mock and quantum modularity.

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