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Conditioned Galton-Watson Trees:

1st & 2nd Moments of their Generation Size

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Abstract

This report gives a brief history and overview of Galton-Watson processes. Then, based largely off of the work of Geiger and Kersting in [4], we construct a conditioned Galton-Watson tree and use this construction to prove new results about the first and second moments of the trees generation sizes. We prove the existence of second moments of the final generation of a conditioned Galton-Watson tree with arbitrary off-spring distribution. We also show strong convergence for the $n-j$th generation size of a conditioned Galton-Watson tree, as well as the existence of first and second moments of the $n-j$th generation.

1 Background

A branching process is a stochastic process that arose as a solution to the problem of the extinction of family names. In 19th century England, despite a booming population, aristocratic surnames were disappearing. Francis Galton and Rev. Henry William Watson together wrote *'On the Probability of the Extinction of Family'* [6] and used branching processes (commonly referred to as the Galton-Watson process[1]) to mathematically describe this phenomenon.

Galton-Watson branching processes are generally defined as Markov chains $\{Z_n : n \geq 0\}$ on the non-negative integers, where Z_n represents the size of the nth generation of a family. Given $p_k \in [0,1]$, $k = 0, 1, ...$, with $\sum_{k=0}^{\infty} p_k = 1$, the process is defined as follows:

We start with one particle, $Z_0 = 1$, unless specified otherwise. It has k children with probability p_k. Then each of these children (if any) also have children with the same progeny (or "offspring") distribution $\{p_k : k \geq 0\}$, independently of one other and of their parent. This process repeats indefinitely, unless a given generation produces no offspring, in which case we say the process is extinct.

More formally,

Definition 1. A Galton-Watson process is a Markov chain $\{Z_n : n = 0, 1, ...\}$ defined as follows:

Let $p_0, p_1, ... \in [0,1]$ such that $\sum_{k=0}^{\infty} p_k = 1$. Let $\{X_{n,i}\}_{n,i\geq 0}$ be a family of independent random variables such that $\mathbb{P}(X_{n,i} = k) = p_k$, for any $k \in \mathbb{N} \cup \{0\}$.

Define $Z_0 = 1$ and

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_{n-1,i}.$$

[1]In fact, the process was first described three decades earlier by French mathematician I.J Bienaymé [3], so it is sometimes referred to as a Bienaymé-Galton-Watson process. There is reasonable argument outlined in [1] to refer to the process only as the Bienaymé process. In this report, we shall stick to 'Galton-Watson process' to maintain consistency with the main papers cited.

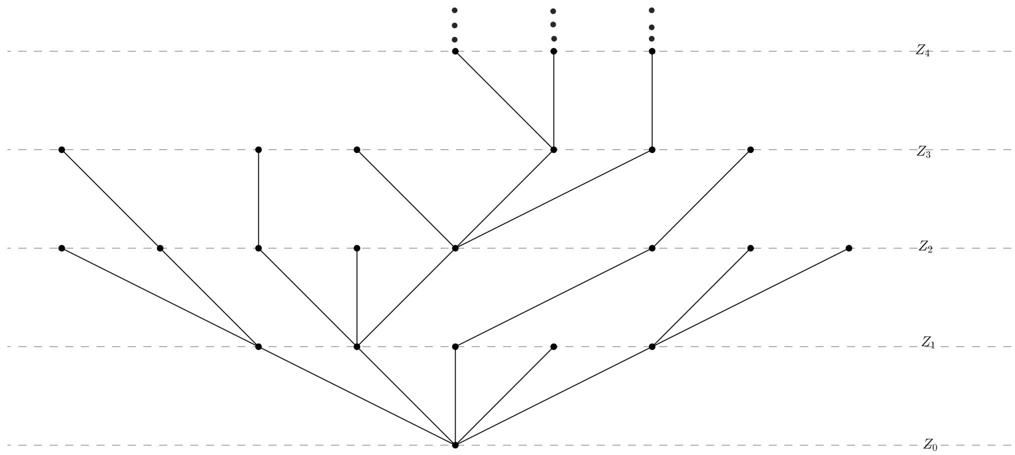

Figure 1: The Galton-Watson process defined above gives rise to a rooted planar tree structure as shown. Each node of the tree represents an individual, and each edge represents a parent-offspring relationship. This tree shows generations 0 to 4 of a Galton-Watson process of unspecified offspring distribution. Notice that individuals in any given generation produce offspring only once, simultaneously, and after discrete time intervals.

2 Moments of the Galton-Watson Process

The probability generating function p.g.f. of the branching process is very useful in its analysis. It is defined as

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \leq 1. \tag{1}$$

The moments of the process, when they exist, can be expressed as derivatives of $f(s)$ at $s = 1$. Call the first moment of the process μ, the expected number of offspring an individual will have:

$$\mu := f'(1) = \mathbb{E}[Z_1] = \sum_{k=1}^{\infty} k p_k. \tag{2}$$

From the definition of f as a power series with non-negative coefficients, we see f is strictly convex and increasing. Also, $f(1) = 1$ and, allowing $0^0 = 1$, we see that $f(0) = p_0$. Thus, define q to be the smallest non-negative root of the equation $f(t) = t, t \in \mathbb{R}$. By the stated properties of f, we know that q exists. Moreover,

Proposition 2.1 (Extinction Criterion)**.** *If $\mu \leq 1$ then $q = 1$ and if $\mu > 1$ then $q \leq 1$. We call q the extinction probability of the branching process i.e. $q = \lim_{n\to\infty} \mathbb{P}(Z_n = 0)$.*

Remark 2.2. *The extinction criterion has one exception, namely when $p_1 = 1$. Then $\mu = 1$, but $q = 0$.*

This situation is illustrated clearly in Fig. 2.

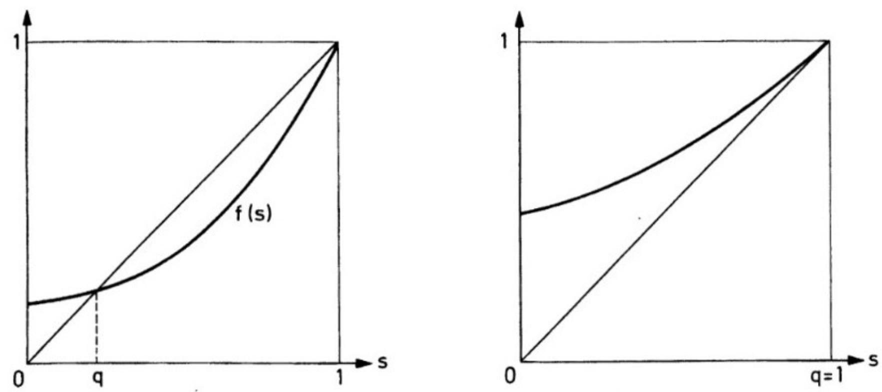

Figure 2: Since $f(s)$ is a convex and strictly increasing function with $f(1) = 1$, if $f'(1) = \mu > 1$, the situation illustrated in the left-hand graph arises. Similarly, in the right-hand graph, if $f'(1) = \mu > 1$, $f(s)$ must approach $(1, 1)$ from above and can only intersect the line $x = y$ at $q = 1$.

Because of Prop.2.1, we call a Galton-Watson process subcritical, critical and supercritical if $\mu < 1, = 1$, and > 1 respectively.

One of the most interesting questions we can ask about a Galton-Watson process is how it evolves over time. The following result shows that there are only two distinct possibilities for the size of the nth generation of any Galton-Watson process as $n \to \infty$.

Proposition 2.3 (Instability of Z_n). *Let $p_1 \neq 1$ and $\mu = \mathbb{E}[Z_1] < \infty$. Then* $\lim_{n\to\infty} \mathbb{P}(Z_n = k) = 0, k = 1, 2, \ldots.$ *Moreover, $Z_n \to \infty$ with probability $1 - q$ and $Z_n \to 0$ with probability q.*

This statement can be proven by showing that the states $k = 1, 2, \ldots$ are transient, as done by Harris in [5].

Remark 2.4. *Again, in the case $p_1 = 1$ each individual will have exactly 1 offspring and hence $Z_n = 1$ for all n.*

The unstable behaviour depicted by Prop.2.3 differs vastly from the usual behaviour we see in biological systems. Typically, a population can grow unhindered and then stabilise to a limit supportable by its environment. However, the basic Galton-Watson process is still useful in events such as the early stages of disease outbreak or family development.

3 Conditioned Galton-Watson trees

We can introduce restrictions onto a Galton-Watson process which impact of the shape and size of the trees. For example, we could introduce an immigration distribution, or allow reproduction to occur over continuous time intervals. These kinds of ammendments to the basic Galton-Watson process are dealt with in [2].

One can also restrict the tree by its height (number of generations), width (size of any generation) or size (number of individuals in the entire tree). This is called conditioning the tree which allows us to ask interesting questions about the shape of the tree and the distribution of generation size.

We will investigate Galton-Watson trees conditioned on their height, i.e. restricting on the number of generations. What follows is largely based off of the work of Geiger and Kersting in their paper '*The Galton-Watson tree conditioned on its height*' [4].

3.1 Growing a conditioned Galton-Watson tree from the top

The following construction is taken almost directly from [4], as it forms the basis of the results of this report.

Let $H(T)$ denote the height of a Galton-Watson tree, i.e. for $n \geq 0$,

$$H(T) = n \iff Z_n > 0, Z_{n+1} = 0.$$

Let $T^{(i)}$, $1 \leq i \leq Z_1(T)$ be the subtrees founded by the first generation individuals of T.

For a finite tree T, define $R(T)$ as

$$R(T) := \min_i \{1 \leq i \leq Z_1(T) | H(T^{(i)}) = H(T) - 1\}$$

i.e. the rank of the left-most subtree $T^{(i)}$ of maximal height.

The following lemma suggests a method to grow a conditioned Galton-Watson tree from the top backwards.

Lemma 3.1. *The subtrees $T^{(i)}, 1 \leq i \leq Z_1$, are conditionally independent given $\{R = j, Z_1 = k, H = n+1\}, 1 \leq j \leq k < \infty, n \geq 0$, with*

$$\mathcal{L}(T^{(i)} | R = j, Z_1 = k, H = n+1) = \begin{cases} \mathcal{L}(T | Z_n = 0), & 1 \leq i \leq j-1; \\ \mathcal{L}(T | H = n), & i = j; \\ \mathcal{L}(T | Z_{n+1} = 0), & j+1 \leq i \leq k. \end{cases}$$

The conditional joint distribution of R and Z_1 is

$$\mathbb{P}(R = j, Z_1 = k | H = n+1) = c_n p_k \mathbb{P}(Z_n = 0)^{j-1} \mathbb{P}(Z_{n+1} = 0)^{k-j}, \tag{3}$$

where $c_n = \mathbb{P}(H = n)/\mathbb{P}(H = n+1), n \geq 0$.

From this conditional independence, we can construct an increasing sequence of trees $(\bar{T}_n)_{n\geq 0}$ such that $\bar{T}_n \stackrel{d}{=} T|H = n$. The tree $\bar{T}_{n+1}$ is obtained from $\bar{T}_n$ by attaching independent subtrees at the bottom of the line of descent of the left-most individual at maximal height.

Construction. Let $(V_{n+1}, W_{n+1}), n \geq 0$, be a sequence of independent random variables with distribution (3),

$$\mathbb{P}(V_{n+1} = j, W_{n+1} = k) = c_n p_k \mathbb{P}(Z_n = 0)^{j-1} \mathbb{P}(Z_{n+1} = 0)^{k-j},$$

and let $\bar{T}_0$ be a Galton-Watson tree of height 0 (i.e. $\bar{T}_0$ is just its root). Inductively construct $\bar{T}_{n+1}, n \geq 0$, as follows (see Fig.3):

- Let the first generation size of $\bar{T}_{n+1}$ be W_{n+1}.
- Let $\bar{T}_n$ be the subtree founded by the V_{n+1}th first generation particle of $\bar{T}_{n+1}$.
- Attach independent Galton-Watson trees conditioned on height strictly less than n to the $V_{n+1} - 1$ siblings to the left of the distinguished first generation particle.
- Attach independent Galton-Watson trees conditioned on height strictly less than $n+1$ to the $W_{n+1} - V_{n+1}$ siblings to the right of the distinguished first generation particle.

The tree $\bar{T}_n$ has the same probabilistic structure as a Galton-Watson tree conditioned on height n, i.e. $\bar{T}_n \stackrel{d}{=} T|H = n$ for any $n \geq 0$.

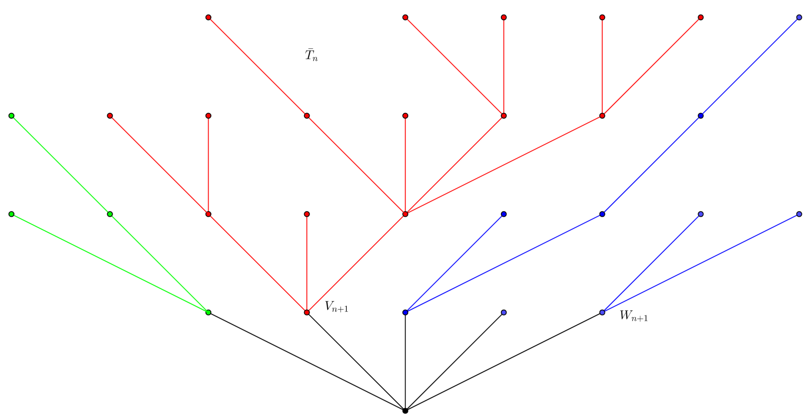

Figure 3: A conditioned Galton-Watson tree constructed as above. The blue lines of descent are conditioned to die out after no more than n generations. The green lines of descent are conditioned to die out after no more than $n-1$ generations. The red subtree, $\bar{T}_n$, is conditioned to have exactly n generations. These distinct subtrees are therefore conditionally independent of one another.

Remark 3.2. *For any offspring distribution* $(p_k)_{k\geq 0}, (V_n, W_n)$ *has a weak limit* (V_∞, W_∞) *with distribution*

$$\mathbb{P}(V_\infty = j, W_\infty = k) = c_\infty p_k q^{k-1}, \;\; 1 \leq j \leq k, \tag{4}$$

where

$$c_\infty := \lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\mathbb{P}(H=n)}{\mathbb{P}(H=n+1)} = (\sum_{k=1}^{\infty} k q^{k-1} p_k)^{-1}. \tag{5}$$

Notice, from this construction, descendandents of the first generation individuals to the left of the distinguished first generation individual do not contribute to the final generation of the tree. Individuals in the final generation of $\bar{T}_{n+1}$ are either individuals in the final generation of $\bar{T}_n$ or descendants of the siblings to the right hand side of the distinguished first generation particle. This leads to the following representation of the final generation size of a Galton-Watson tree conditioned on its height:

Let $Z_{n,i}$ and X_{n+1}, $n \geq 0, i \geq 1$, be independent random variables where

$$Z_{n,i} \stackrel{d}{=} Z_n | Z_{n+1} = 0 \tag{6}$$

and $X_{n+1} \stackrel{d}{=} W_{n+1} - V_{n+1}$ i.e.

$$\mathbb{P}(X_{n+1} = k) = c_n \mathbb{P}(Z_{n+1} = 0)^k \sum_{j=k+1}^{\infty} p_j \mathbb{P}(Z_n = 0)^{j-(k+1)}, \;\; k \geq 0. \tag{7}$$

Then define $\bar{Z}_0 := 1$ and

$$\bar{Z}_{n+1} := \bar{Z}_n + \sum_{i=1}^{X_{n+1}} Z_{n,i}, \;\; n \geq 0. \tag{8}$$

Similarly, for some fixed j, $0 \leq j \leq n$, we have a representation of the $n-j$th generation size of a Galton-Watson tree conditioned on height n. In this case, siblings to the left of the distinguished first generation particle can contribute descendants to the $n-j$th generation. Thus we define new random variables to account for the 'left-hand' and 'right-hand' side of the tree, independent of one another:

Let $Q_{j,i}$ and $Y_{j,i}$ be independent random variables with the following laws:

$$Q_{j,i} \stackrel{d}{=} Z_{n-j} | Z_{n+1} = 0 \tag{9}$$

$$Y_{j,i} \stackrel{d}{=} Z_{n-j} | Z_n = 0. \tag{10}$$

Then for some fixed j with $0 \leq j \leq n$, define $\bar{Z}_j^{(j)} = 1$ and

$$\bar{Z}_{n+1}^{(j)} := \bar{Z}_n^{(j)} + \sum_{i=1}^{V_{n+1}-1} Y_{j,i} + \sum_{u=1}^{X_{n+1}} Q_{j,i} \tag{11}$$

4 First and Second Moments of Generation Size

The following section shows existence of the second moment of the final generation size, as well as the strong convergence and existence of first and second moments of the jth generation size back from the final generation. These novel results require the following result for the final generation size of a conditioned Galton-Watson tree as stated by Geiger & Kersting [4], and proven using the above construction for a conditioned tree:

Proposition 4.1. *Suppose* $0 < p_0 < 1$. *Then*

$$\bar{Z}_n \xrightarrow{a.s.} \bar{Z}_\infty \;\; as \;\; n \to \infty, \tag{12}$$

$$\mathbb{E}[\bar{Z}_\infty] = \lim_{n\to\infty} \mathbb{E}[\bar{Z}_n] < \infty. \tag{13}$$

Using similar methods, this result can be extended to the $n - j$th generation of a conditioned Galton-Watson tree, as well as to second moments for both cases. Notice that the results hold for any arbitrary off-spring distribution.

The following equations are derived by Geiger and Kersting in [4], and are used repeatedly in the subsequent proofs, and so are stated here for reference:

$$\mathbb{E}[X_{n+1}] \leq \sum_{j=1}^{\infty} j^2 p_j \mathbb{P}(Z_{n+1} = 0)^{j-1}, \tag{14}$$

$$\mathbb{P}(Z_n > 0 | Z_{n+1} = 0) \asymp \mathbb{P}(H = n) \asymp \mathbb{P}(H = n + 1). \tag{15}$$

Similarly, the following statement is used in the subsequent proofs, and is derived in [4]:

Let $(a_n)_{n\geq 0}$ be an increasing sequence. Then

$$\sum_{n=0}^{\infty} a_n^{-1}(a_{n+1} - a_n) < \infty \iff \lim_{n\to\infty} a_n < \infty. \tag{16}$$

4.1 First Moment of the $n - j$th Generation

Proposition 4.2. *Suppose* $0 < p_0 < 1$. *Then for any fixed* $j \in \mathbb{N}$ *such that* $0 \leq j \leq n$

$$\bar{Z}_n^{(j)} \xrightarrow{a.s.} \bar{Z}_\infty^{(j)} \;\; as \;\; n \to \infty$$

$$\mathbb{E}[\bar{Z}_\infty^{(j)}] = \lim_{n\to\infty} \mathbb{E}[\bar{Z}_n^{(j)}] < \infty.$$

Proof. By definition, $(\bar{Z}_n^{(j)})_{n\geq 0}$ is an increasing sequence and so has an almost sure limit which we denote as $\bar{Z}_\infty^{(j)}$. By the Monotone Convergence Theorem, $\lim_{n\to\infty}\mathbb{E}[\bar{Z}_n^{(j)}] = \mathbb{E}[\bar{Z}_\infty^{(j)}] \leq \infty$. The finite condition follows by induction on j.

The case when for $\mathbb{E}[\bar{Z}_\infty^{(0)}]$ is precisely the result of Prop.4.1.

Assume that the result holds for all $i < j \in \mathbb{N}$, i.e.

$$\lim_{n\to\infty}\mathbb{E}[\bar{Z}_n^{(i)}] < \infty \quad \forall\, i < j \in \mathbb{N}. \tag{17}$$

Then, using the independence of $V_{n+1}, Y_{j,i}$ and $X_{n+1}, Q_{j,i}$,

$$\begin{aligned}\mathbb{E}[\overline{Z}_{n+1}^{(j)}] - \mathbb{E}[\overline{Z}_n^{(j)}] &= \mathbb{E}\left[\sum_{i=1}^{V_{n+1}-1} Y_{j,i}\right] + \mathbb{E}\left[\sum_{u=1}^{X_{n+1}} Q_{j,u}\right]\\ &= \mathbb{E}[V_{n+1}-1]\mathbb{E}[Z_{n-j}|Z_n=0] + \mathbb{E}[X_{n+1}]\mathbb{E}[Z_{n-j}|Z_{n+1}=0].\end{aligned} \tag{18}$$

Now

$$\begin{aligned}\mathbb{E}[Z_{n-j}|Z_n=0] &= \mathbb{E}\left[Z_{n-j}\sum_{i=0}^{j}\mathbb{1}_{Z_{n-i-1}>0,Z_{n-i}=0}|Z_n=0\right]\\ &= \sum_{i=0}^{j}\mathbb{E}[Z_{n-j}|Z_{n-i-1}>0,Z_{n-i}=0]\mathbb{P}(Z_{n-i-1}>0,Z_{n-i}=0|Z_n=0)\\ &= \sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-1-i}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0),\end{aligned} \tag{19}$$

and similarly

$$\begin{aligned}\mathbb{E}[Z_{n-j}|Z_{n+1}=0] &= \mathbb{E}\left[Z_{n-j}\sum_{i=0}^{j}\mathbb{1}_{Z_{n-i}>0,Z_{n-i+1}=0}|Z_{n+1}=0\right]\\ &= \sum_{i=0}^{j}\mathbb{E}[Z_{n-j}|Z_{n-i}>0,Z_{n-i+1}=0]\mathbb{P}(Z_{n-i}>0,Z_{n-i+1}=0|Z_{n+1}=0)\\ &= \sum_{i=0}^{j}\mathbb{E}[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0).\end{aligned} \tag{20}$$

By (16), as used in [4], (18)

$$\mathbb{E}[\bar{Z}_\infty^{(j)}] < \infty \iff \sum_{n=0}^{\infty}\mathbb{E}[\bar{Z}_n^{(j)}]^{-1}(\mathbb{E}[\overline{Z}_{n+1}^{(j)}] - \mathbb{E}[\overline{Z}_n^{(j)}]) < \infty$$

which holds if any only if the following three equations hold:

$$\sum_{n=0}^{\infty}\mathbb{E}[\bar{Z}_n^{(j)}]^{-1}\mathbb{E}[V_{n+1}-1]\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-1-i}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)<\infty, \quad (21)$$

$$\sum_{n=0}^{\infty}\mathbb{E}[\bar{Z}_n^{(j)}]^{-1}\mathbb{E}[X_{n+1}]\sum_{i=1}^{j}\mathbb{E}[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0)<\infty, \quad (22)$$

and

$$\sum_{n=0}^{\infty}\mathbb{E}[X_{n+1}]\mathbb{P}(Z_n>0|Z_{n+1}=0)<\infty. \quad (23)$$

We begin by finding upper bounds for unknown terms in the above expressions. Clearly, $\mathbb{E}[\bar{Z}_n^{(j)}]\geq 1$ for all n, and so $\mathbb{E}[\bar{Z}_n^{(j)}]^{-1}\leq 1$ for all n. Next,

$$\begin{aligned}\sum_{i=1}^{j}\mathbb{E}[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0) &\leq \sum_{i=1}^{j}\mathbb{E}[\bar{Z}_{\infty}^{(j-i)}]\frac{\mathbb{P}(H=n-i)}{\mathbb{P}(Z_{n-i+1}=0)}\\ &\overset{(17)}{\asymp}\sum_{i=1}^{j}\frac{\mathbb{P}(H=n-i)}{\mathbb{P}(Z_{n-i+1}=0)}\\ &\asymp\sum_{i=1}^{j}\mathbb{P}(H=n-i)\\ &\asymp\sum_{i=1}^{j}\mathbb{P}(H=n+1)\\ &\asymp\mathbb{P}(H=n+1). \quad (24)\end{aligned}$$

Notice that this expression is asymptotically bounded by a term which depends only on n. Similarly,

$$\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)\asymp\mathbb{P}(H=n)\asymp\mathbb{P}(H=n+1). \quad (25)$$

Finally, we find an upper bound for $\mathbb{E}[V_{n+1}-1]$.

$$\begin{aligned}\mathbb{E}[V_{n+1}-1] &= \sum_{j=1}^{\infty}j\mathbb{P}(V_{n+1}-1=j)\\ &= \sum_{j=1}^{\infty}j\sum_{k=1}^{\infty}\mathbb{P}(V_{n+1}=j+1,W_{n+1}=k)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} j \sum_{k=j+1}^{\infty} c_n p_k \mathbb{P}(Z_n = 0)^j \mathbb{P}(Z_{n+1} = 0)^{k-(j+1)} \\
&\asymp \sum_{j=1}^{\infty} j \mathbb{P}(Z_n = 0)^j \sum_{k=j+1}^{\infty} p_k \mathbb{P}(Z_{n+1} = 0)^{k-(j+1)} \\
&\leq \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} j p_k \mathbb{P}(Z_{n+1} = 0)^{k-1} \\
&\leq \sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}. \qquad (26)
\end{aligned}
$$

Now, an upper bound for (23) is found in [4], where c denotes a positive constant, as

$$
\sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}] \mathbb{P}(Z_n > 0 | Z_{n+1} = 0) \overset{(14),(15)}{\leq} c \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k < \infty. \qquad (27)
$$

Next, (22) can be bounded as follows:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^{(j)}]^{-1} \mathbb{E}[X_{n+1}] \sum_{i=1}^{j} \mathbb{E}[\bar{Z}_{n-i}^{(j-i)}] \mathbb{P}(Z_{n-i} > 0 | Z_{n-i+1} = 0) \mathbb{P}(Z_{n-i+1} = 0 | Z_{n+1} = 0) \\
&\qquad \overset{(24)}{\asymp} \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}] \mathbb{P}(H = n+1) \\
&\qquad \leq c \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1} \mathbb{P}(H = n+1) \\
&\qquad \asymp \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k < \infty. \qquad (28)
\end{aligned}
$$

And finally (21) can be bounded as follows, using (25), (26) and the Binomial Theorem,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^{(j)}]^{-1} \mathbb{E}[V_{n+1} - 1] \sum_{i=0}^{j-1} \mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}] \mathbb{P}(Z_{n-1-i} > 0 | Z_{n-i} = 0) \mathbb{P}(Z_{n-i} = 0 | Z_n = 0) \\
&\qquad \leq \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1} \mathbb{P}(H = n+1) \\
&\qquad \asymp \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k < \infty. \qquad (29)
\end{aligned}
$$

Thus, the claim follows.

□

4.2 Second Moment of the nth Generation

Proposition 4.3. *Suppose* $0 < p_0 < 1$. *Then*

$$(\bar{Z}_n)^2 \xrightarrow{a.s.} (\bar{Z}_\infty)^2 \text{ as } n \to \infty$$
$$\mathbb{E}[(\bar{Z}_\infty)^2] = \lim_{n\to\infty} \mathbb{E}[(\bar{Z}_n)^2] < \infty.$$

Proof. From Prop.4.1, we know that $\bar{Z}_n \xrightarrow{a.s.} \bar{Z}_\infty$ as $n \to \infty$. Thus $(\bar{Z}_n)^2 \xrightarrow{a.s.} (\bar{Z}_\infty)^2$ as $n \to \infty$. By definition $((\bar{Z}_n)^2)_{n\geq 0}$ is an increasing sequence. Hence, by the Monotone Convergence Theorem, $\lim_{n\to\infty} \mathbb{E}[(\bar{Z}_n)^2] == \mathbb{E}[\lim_{n\to\infty}(\bar{Z}_n)^2] = \mathbb{E}[(\bar{Z}_\infty)^2] \leq \infty$.

Then, by the independence of X_{n+1} and $Z_{n,i}$, we have

$$\begin{aligned}
\mathbb{E}[(\bar{Z}_{n+1})^2] - \mathbb{E}[(\bar{Z}_n)^2] &= 2\mathbb{E}\left[\bar{Z}_n \sum_{i=1}^{X_{n+1}} Z_{n,i}\right] + \mathbb{E}\left[(\sum_{i=1}^{X_{n+1}} Z_{n,i})^2\right] \\
&= 2\mathbb{E}\left[\bar{Z}_n \sum_{i=1}^{X_{n+1}} Z_{n,i}\right] + \mathbb{E}\left[\sum_{i=1}^{X_{n+1}} Z_{n,i}^2\right] + 2\mathbb{E}\left[\sum_{\substack{i<j \\ j=1}}^{X_{n+1}} Z_{n,i}Z_{n,j}\right] \\
&= 2\mathbb{E}[\bar{Z}_n]\mathbb{E}[X_{n+1}]\mathbb{E}[Z_n|\bar{Z}_{n+1} = 0] + \mathbb{E}[X_{n+1}]\mathbb{E}[Z_n^2|Z_{n+1} = 0] \\
&\quad + \mathbb{E}[X_{n+1}(X_{n+1} - 1)]\mathbb{E}[Z_n|Z_{n+1} = 0]^2 \\
&= 2\mathbb{E}[\bar{Z}_n]^2\mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0) + \mathbb{E}[X_{n+1}]\mathbb{E}[\bar{Z}_n^2]\mathbb{P}(Z_n|Z_{n+1} = 0) \\
&\quad + \mathbb{E}[(X_{n+1})^2]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 \\
&\quad - \mathbb{E}[X_{n+1}]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2.
\end{aligned} \tag{30}$$

By (16) from [4],

$$\mathbb{E}[(\bar{Z}_\infty)^2] < \infty \iff \sum_{n=0}^{\infty} \mathbb{E}[(\bar{Z}_n)^2]^{-1}(\mathbb{E}[(\bar{Z}_{n+1})^2] - \mathbb{E}[(\bar{Z}_n)^2]) < \infty$$

which holds if and only if the following four equations hold:

$$\sum_{n=0}^{\infty} 2\mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[\bar{Z}_n]^2\mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0) < \infty, \tag{31}$$

$$\sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]P(Z_n|Z_{n+1} = 0) < \infty, \tag{32}$$

$$\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[(X_{n+1})^2]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 < \infty \tag{33}$$

and

$$\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[X_{n+1}]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 < \infty. \tag{34}$$

We first bound any unknown terms in the above expressions. Since $\mathbb{E}[\bar{Z}_n^2] \geq 1$ for all n, we have $\mathbb{E}[\bar{Z}_n^2]^{-1} \leq 1$ for all n. Also, by Prop.4.1,

$$\mathbb{E}[\bar{Z}_n] \leq \mathbb{E}[\bar{Z}_\infty] = A < \infty \; \forall \, n. \tag{35}$$

Lastly, (33) requires a bound for $\mathbb{E}[(X_{n+1}^2)]$:

$$\begin{aligned}
\mathbb{E}[(X_{n+1})^2] &\asymp \sum_{k=1}^{\infty} k^2 \mathbb{P}(Z_{n+1} = 0)^k \sum_{j=k+1}^{\infty} p_j \mathbb{P}(Z_n = 0)^{j-(k+1)} \\
&\leq \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} k^2 p_j \mathbb{P}(Z_{n+1} = 0)^{j-1} \\
&\leq \sum_{j=1}^{\infty} j^3 p_j \mathbb{P}(Z_{n+1} = 0)^{j-1},
\end{aligned} \tag{36}$$

Then (31) is bounded above as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2\mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[\bar{Z}_n]^2\mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0) &\overset{(35)}{\leq} A^2 \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0) \\
&\overset{(27)}{<} cA^2 \sum_{k=1}^{\infty} kp_k \mathbb{P}(H < \infty)^k < \infty.
\end{aligned} \tag{37}$$

Equation (32) follows directly from (27).

Equation (33) can be bounded above using the Binomial Theorem:

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[(X_{n+1})^2]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 &\overset{(35)}{\leq} A^2 \sum_{n=0}^{\infty} \mathbb{E}[(X_{n+1})^2]\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 \\
&\overset{(36),(15)}{\leq} A^2 \sum_{n=0}^{\infty}\sum_{j=1}^{\infty} j^3 p_j \mathbb{P}(Z_{n+1} = 0)^{j-1}\mathbb{P}(H = n+1)^2 \\
&= A^2 \sum_{j=1}^{\infty} jp_j \sum_{n=0}^{\infty} j^2 \mathbb{P}(H \leq n)^{j-1}\mathbb{P}(H = n+1)^2 \\
&\leq A^2 \sum_{j=1}^{\infty} jp_j \sum_{n=0}^{\infty} \mathbb{P}(H \leq n+1)^{j+1} - \mathbb{P}(H \leq n)^{j+1}
\end{aligned}$$

$$\leq A^2 \sum_{j=1}^{\infty} j p_j \mathbb{P}(H < \infty)^{j+1} < \infty. \tag{38}$$

Finally, (34) follows from (27) and (35)

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}[\bar{Z}_n^2]^{-1}\mathbb{E}[X_{n+1}]\mathbb{E}[\bar{Z}_n]^2\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 &\leq A^2 \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0)^2 \\
&\leq A^2 \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]\mathbb{P}(Z_n > 0|Z_{n+1} = 0) \\
&\overset{(27)}{<} \infty. \qquad (39)
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} \mathbb{E}[(\bar{Z}_n)^2]^{-1}(\mathbb{E}[(\bar{Z}_{n+1})^2] - \mathbb{E}[(\bar{Z}_n)^2]) < \infty$ and the result follows. □

4.3 Second Moment of the $n - j$th Generation

Proposition 4.4. *Suppose* $0 < p_0 < 1$*. Then for any fixed* $j \in \mathbb{N}$ *such that* $0 \leq j \leq n$

$$(\bar{Z}_n^{(j)})^2 \xrightarrow{a.s.} (\bar{Z}_\infty^{(j)})^2$$
$$\mathbb{E}[(\bar{Z}_\infty^{(j)})^2] = \lim_{n\to\infty} \mathbb{E}[(\bar{Z}_n^{(j)})^2] < \infty.$$

Proof. We know from Prop.4.2 that $\bar{Z}_n^{(j)} \xrightarrow{a.s.} \bar{Z}_\infty^{(j)}$ Thus $(\bar{Z}_n^{(j)})^2 \xrightarrow{a.s.} (\bar{Z}_\infty^{(j)})^2$. By definition, $((\bar{Z}_n^{(j)})^2)_{n\geq 0}$ is an increasing sequence. By the Monotone Convergence Theorem, $\lim_{n\to\infty} \mathbb{E}[(\bar{Z}_n^{(j)})^2] = \mathbb{E}[\lim_{n\to\infty}(\bar{Z}_n^{(j)})^2] = \mathbb{E}[(\bar{Z}_\infty^{(j)})^2] \leq \infty$. The finite condition follows from induction on j.

The case $\mathbb{E}[(\bar{Z}_n^{(0)})^2]$ is exactly the result of Prop.4.3.

Assume that the result holds for all $i < j \in \mathbb{N}$ i.e.

$$\lim_{n\to\infty} \mathbb{E}[(\bar{Z}_n^{(i)})^2] < \infty \; \forall \, i < j \in \mathbb{N}. \tag{40}$$

Then, using the independence of $V_{n+1}, Y_{j,i}$ and $X_{n+1}, Q_{j,i}$, we have

$$\begin{aligned}
\mathbb{E}[(\bar{Z}_{n+1}^{(j)})^2] - \mathbb{E}[(\bar{Z}_n^{(j)})^2] =& \mathbb{E}\left[(\sum_{i=1}^{V_{n+1}-1} Y_{j,i})^2\right] + \mathbb{E}\left[(\sum_{u=1}^{X_{n+1}} Q_{j,u})^2\right] \\
&+ 2\mathbb{E}\left[\bar{Z}_n^{(j)} \sum_{i=1}^{V_{n+1}-1} Y_{j,i}\right] + 2\mathbb{E}\left[\bar{Z}_n^{(j)} \sum_{u=1}^{X_{n+1}} Q_{j,u}\right] + 2\mathbb{E}\left[\sum_{i=1}^{V_{n+1}-1} Y_{j,i} \sum_{u=1}^{X_{n+1}} Q_{j,u}\right]
\end{aligned}$$

where

$$\mathbb{E}\left[(\sum_{i=1}^{V_{n+1}-1} Y_{j,i})^2\right] = \mathbb{E}[V_{n+1}-1]\sum_{i=0}^{j-1}\mathbb{E}[(\bar{Z}_{n-1-i}^{(j-1-i)})^2]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)$$
$$+\mathbb{E}[(V_{n+1}-1)(V_{n+1}-2)](\sum_{i=0}^{j-1} E[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}|Z_n=0))^2, \tag{41}$$

$$\mathbb{E}\left[(\sum_{u=1}^{X_{n+1}} Q_{j,u})^2\right] = \mathbb{E}[X_{n+1}]\sum_{i=0}^{j}\mathbb{E}[(\bar{Z}_{n-i}^{(j-i)})^2]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0)$$
$$+\mathbb{E}[X_{n+1}(X_{n+1}-1)](\sum_{i=0}^{j}\mathbb{E}[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0))^2, \tag{42}$$

$$\mathbb{E}\left[\bar{Z}_n^{(j)}\sum_{i=1}^{V_{n+1}-1} Y_{j,i}\right]$$
$$= \mathbb{E}[\bar{Z}_n^{(j)}]\mathbb{E}[V_{n+1}-1]\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0), \tag{43}$$

$$\mathbb{E}\left[\bar{Z}_n^{(j)}\sum_{u=1}^{X_{n+1}} Q_{j,u}\right]$$
$$= \mathbb{E}[\bar{Z}_n^{(j)}]\mathbb{E}[X_{n+1}]\sum_{i=0}^{j}\mathbb{E}[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i}>0|Z_{n-i+1}=0)\mathbb{P}(Z_{n-i+1}=0|Z_{n+1}=0), \tag{44}$$

and

$$\mathbb{E}\left[\sum_{i=1}^{V_{n+1}-1} Y_{j,i}\sum_{u=1}^{X_{n+1}} Q_{j,u}\right] = \mathbb{E}[(V_{n+1}-1)X_{n+1}]$$
$$\times\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)$$
$$\times\sum_{k=0}^{j}\mathbb{E}[\bar{Z}_{n-k}^{(j-k)}]\mathbb{P}(Z_{n-k}>0|Z_{n-k+1}=0)\mathbb{P}(Z_{n-k+1}=0|Z_{n+1}=0). \tag{45}$$

To prove the claim we once more use (16):

$$\mathbb{E}[(\bar{Z}_\infty^{(j)})^2] < \infty \iff \sum_{n=0}^{\infty} \mathbb{E}[(\bar{Z}_n^{(j)})^2]^{-1}(\mathbb{E}[(\bar{Z}_{n+1}^{(j)})^2] - \mathbb{E}[(\bar{Z}_n^{(j)})^2]) < \infty \tag{46}$$

which holds if and only if each of (41) − (45), summed from $n = 0$ to ∞, are finite.

We start by bounding any unknown terms above. Firstly, $\mathbb{E}[(\bar{Z}_n^{(j)})^2]^{-1} \leq 1$ for all n. Then from Prop.4.2, we have

$$\mathbb{E}[\bar{Z}_n^{(j)}] \leq \mathbb{E}[\bar{Z}_\infty^{(j)}] = B < \infty. \tag{47}$$

Next,

$$\begin{aligned}
\sum_{i=1}^{j} \mathbb{E}[(\bar{Z}_{n-i}^{(j-i)})^2]\mathbb{P}(Z_{n-i} > 0|Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0|Z_{n+1} = 0) &\leq \sum_{i=1}^{j} \mathbb{E}[(\bar{Z}_\infty^{(j-i)})^2]\frac{\mathbb{P}(H = n-i)}{\mathbb{P}(Z_{n-i+1} = 0)} \\
&\overset{(40)}{\asymp} \sum_{i=1}^{j} \frac{\mathbb{P}(H = n-i)}{\mathbb{P}(Z_{n-i+1} = 0)} \\
&\asymp \sum_{i=1}^{j} \mathbb{P}(H = n-i) \\
&\asymp \sum_{i=1}^{j} \mathbb{P}(H = n) \\
&\asymp \mathbb{P}(H = n) \\
&\asymp \mathbb{P}(H = n+1).
\end{aligned} \tag{48}$$

Similarly,

$$\sum_{i=0}^{j-1} \mathbb{E}[(\bar{Z}_{n-1-i}^{(j-1-i)})^2]\mathbb{P}(Z_{n-i-1} > 0|Z_{n-i} = 0)\mathbb{P}(Z_{n-i} = 0|Z_n = 0) \asymp \mathbb{P}(H = n) \asymp \mathbb{P}(H = n+1). \tag{49}$$

Next, for (41), we require an upper bound for $\mathbb{E}[(V_{n+1} - 1)^2]$:

$$\begin{aligned}
\mathbb{E}[(V_{n+1} - 1)^2] &= \sum_{j=1}^{\infty} j^2 \mathbb{P}(V_{n+1} - 1 = j) \\
&= \sum_{j=1}^{\infty} j^2 \sum_{k=j+1}^{\infty} \mathbb{P}(V_{n+1} = j+1, W_{n+1} = k) \\
&= \sum_{j=1}^{\infty} j^2 \sum_{k=j+1}^{\infty} c_n p_k \mathbb{P}(Z_n = 0)^j \mathbb{P}(Z_{n+1} = 0)^{k-(j+1)} \\
&\asymp \sum_{j=1}^{\infty} j^2 \mathbb{P}(Z_n = 0)^j \sum_{k=j+1}^{\infty} p_k \mathbb{P}(Z_{n+1} = 0)^{k-(j+1)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} j^2 p_k \mathbb{P}(Z_{n+1}=0)^{k-1} \\
&\leq \sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1}=0)^{k-1}. \qquad (50)
\end{aligned}
$$

Lastly, an upper bound for $\mathbb{E}[(V_{n+1}-1)X_{n+1}]$ can be realised in the following way:

$$
\begin{aligned}
\mathbb{E}[(V_{n+1}-1)X_{n+1}] &= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} j(k-j-1)\mathbb{P}(V_{n+1}-1=j, X_{n+1}=k-j-1) \\
&= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} j(k-j-1)\mathbb{P}(V_{n+1}=j+1, W_{n+1}=k) \\
&= \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} j(k-j-1)c_n p_k \mathbb{P}(Z_n=0)^j \mathbb{P}(Z_{n+1}=0)^{k-(j+1)} \\
&\leq \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} jkc_n p_k \mathbb{P}(Z_n=0)^j \mathbb{P}(Z_{n+1}=0)^{k-(j+1)} \\
&\asymp \sum_{j=1}^{\infty} j\mathbb{P}(Z_n=0)^j \sum_{k=j+1}^{\infty} kp_k \mathbb{P}(Z_{n+1}=0)^{k-(j+1)} \\
&\leq \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} jkp_k \mathbb{P}(Z_{n+1}=0)^{k-1} \\
&\leq \sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1}=0)^{k-1}. \qquad (51)
\end{aligned}
$$

Then (41) can be bounded above as

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathbb{E}\left[(\sum_{i=1}^{V_{n+1}-1} Y_{j,i})^2\right] = &\sum_{n=0}^{\infty}\mathbb{E}[V_{n+1}-1] \\
&\times \sum_{i=0}^{j-1}\mathbb{E}[(\bar{Z}_{n-1-i}^{(j-1-i)})^2]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0) \\
&+\sum_{n=0}^{\infty}\mathbb{E}[(V_{n+1}-1)^2] \\
&\times (\sum_{i=0}^{j-1} E[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0))^2 \\
&-\sum_{n=0}^{\infty}\mathbb{E}[V_{n+1}-1]
\end{aligned}
$$

$$\times (\sum_{i=0}^{j-1} E[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1} > 0 | Z_{n-i} = 0)\mathbb{P}(Z_{n-i} = 0 | Z_n = 0))^2$$

$$\overset{(25),(26),(47),(49),(48)}{\leq} \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)$$

$$+ B^2 \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2$$

$$- B^2 \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2$$

$$\leq \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)$$

$$+ B^2 \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2$$

$$- B^2 \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)$$

$$\asymp \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)$$

$$+ \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2$$

$$\overset{(29),(38)}{\leq} \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k + \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^{k+1} < \infty. \tag{52}$$

Next an upper bound for (42) can be found by (14), (24), (36), (48), and the Binomial Theorem:

$$\sum_{n=0}^{\infty} \mathbb{E}\left[(\sum_{u=1}^{X_{n+1}} Q_{j,u})^2 \right] = \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]$$

$$\times \sum_{i=0}^{j} \mathbb{E}[(\bar{Z}_{n-i}^{(j-i)})^2]\mathbb{P}(Z_{n-i} > 0 | Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0 | Z_{n+1} = 0)$$

$$+ \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}(X_{n+1} - 1)]$$

$$\times (\sum_{i=0}^{j} E[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i} > 0 | Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0 | Z_{n+1} = 0))^2$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}]$$

$$
\begin{aligned}
&\times \sum_{i=0}^{j} \mathbb{E}[(\bar{Z}_{n-i}^{(j-i)})^2]\mathbb{P}(Z_{n-i} > 0|Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0|Z_{n+1} = 0) \\
&+ \sum_{n=0}^{\infty} \mathbb{E}[(X_{n+1})^2] \\
&\quad \times (\sum_{i=0}^{j} E[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i} > 0|Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0|Z_{n+1} = 0))^2 \\
&- \sum_{n=0}^{\infty} \mathbb{E}[(X_{n+1})] \\
&\quad \times (\sum_{i=0}^{j} E[\bar{Z}_{n-i}^{(j-i)}]\mathbb{P}(Z_{n-i} > 0|Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0|Z_{n+1} = 0))^2 \\
\le& \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1) \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2 \\
&\quad - \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^2 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1) \\
=& \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} k^3 p_k \mathbb{P}(Z_{n+1} = 0)^{k-1}\mathbb{P}(H = n+1)^2 \\
\le& \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^{k+1} < \infty. \qquad (53)
\end{aligned}
$$

Upper bounds for both (43), (44) follow directly from the proof of (4.2):

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}[\sum_{i=1}^{V_{n+1}-1} Y_{j,i}] &= \sum_{n=0}^{\infty} \mathbb{E}[V_{n+1} - 1] \sum_{i=0}^{j-1} \mathbb{E}[\bar{Z}_{n-1-i}^{j-1-i}]\mathbb{P}(Z_{n-i-1} > 0|Z_{n-i} = 0)\mathbb{P}(Z_{n-i} = 0|Z_n = 0) \\
&\overset{(29)}{\le} \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k < \infty, \qquad (54)
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathbb{E}[\sum_{u=1}^{X_{n+1}} Q_{j,u}] &= \sum_{n=0}^{\infty} \mathbb{E}[X_{n+1}] \sum_{i=0}^{j} \mathbb{E}[\bar{Z}_{n-i}^{j-i}]\mathbb{P}(Z_{n-i} > 0|Z_{n-i+1} = 0)\mathbb{P}(Z_{n-i+1} = 0|Z_{n+1} = 0) \\
&\overset{(28)}{\asymp} \sum_{k=1}^{\infty} k p_k \mathbb{P}(H < \infty)^k < \infty. \qquad (55)
\end{aligned}
$$

Finally, (45) can be bounded above as follows:

$$\sum_{n=0}^{\infty}\mathbb{E}[\sum_{i=1}^{V_{n+1}-1}Y_{j,i}\sum_{u=1}^{X_{n+1}}Q_{j,u}] = \sum_{n=0}^{\infty}\mathbb{E}[(V_{n+1}-1)X_{n+1}]$$

$$\times\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)$$

$$\times\sum_{k=0}^{j}\mathbb{E}[\bar{Z}_{n-k}^{(j-k)}]\mathbb{P}(Z_{n-k}>0|Z_{n-k+1}=0)\mathbb{P}(Z_{n-k+1}=0|Z_{n+1}=0)$$

$$\overset{(51)}{\leq}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m^3p_m\mathbb{P}(Z_{n+1}=0)^{m-1}$$

$$\times\sum_{i=0}^{j-1}\mathbb{E}[\bar{Z}_{n-1-i}^{(j-1-i)}]\mathbb{P}(Z_{n-i-1}>0|Z_{n-i}=0)\mathbb{P}(Z_{n-i}=0|Z_n=0)$$

$$\times\sum_{k=0}^{j}\mathbb{E}[\bar{Z}_{n-k}^{(j-k)}]\mathbb{P}(Z_{n-k}>0|Z_{n-k+1}=0)\mathbb{P}(Z_{n-k+1}=0|Z_{n+1}=0)$$

$$\overset{(24),(25)}{\asymp}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m^3p_m\mathbb{P}(Z_{n+1}=0)^{m-1}\mathbb{P}(H=n+1)^2$$

$$=\sum_{m=1}^{\infty}mp_m\sum_{n=0}^{\infty}m^2\mathbb{P}(Z_{n+1}=0)^{m-1}\mathbb{P}(H=n+1)^2$$

$$=\sum_{m=1}^{\infty}mp_m\sum_{n=0}^{\infty}m^2\mathbb{P}(H\leq n)^{m-1}\mathbb{P}(H=n+1)^2$$

$$\leq\sum_{m=1}^{\infty}mp_m\sum_{n=0}^{\infty}\mathbb{P}(H\leq n+1)^{m+1}-\mathbb{P}(H\leq n)^{m+1}$$

$$\leq\sum_{m=1}^{\infty}mp_m\mathbb{P}(H<\infty)^{m+1}<\infty. \tag{56}$$

Hence, $\mathbb{E}[(\bar{Z}_{\infty}^{(j)})^2]<\infty$ by (46). □

Corollary 4.5. *Let $0<p_0<1$. For some fixed $j,k\in\mathbb{N}$*

$$Cov(\bar{Z}_{\infty}^{(k)},\bar{Z}_{\infty}^{(j)})=\mathbb{E}[\bar{Z}_{\infty}^{(k)}\bar{Z}_{\infty}^{(j)}]-\mathbb{E}[\bar{Z}_{\infty}^{(k)}]\mathbb{E}[\bar{Z}_{\infty}^{(j)}]<\infty. \tag{57}$$

Proof. This follows immediately from Prop 4.4. □

5 Further Study

We have shown that for any arbitrary off-spring distribution, the first and second moments of the final generation of a conditioned Galton-Watson tree exist, as well as the first and second moments of the $n - j$th generation size of a Galton-Watson tree conditioned to have height n $(0 \leq j \leq n)$.

Using these results, and the construction given above, we can continue to investigate properties of the conditioned Galton-Watson tree, including:

- Can we show that the kth moment of the final generation size exists?
- Can we find an explicit expression for $Cov(\bar{Z}_\infty^{(j)}, \bar{Z}_\infty^{(k)})$?
- Can we show convergence of $\mathbb{E}[\bar{Z}_\infty^{(k)}]$ as $k \to \infty$?

On the final question, we state the following conjecture:

Conjecture 5.1. *If* $\mathbb{E}[Z_1^2] < \infty$ *then* $\mathbb{E}[\bar{Z}_\infty^{(k)}]$ *converges as* $k \to \infty$*.*

Notice that Prop 4.1-4.4 hold for any arbitrary off-spring distribution. To show convergence of $E[\bar{Z}_\infty^{(k)}]$ as $k \to \infty$ we require the off-spring distribution to have finite variance.
Otherwise, $\mathbb{E}[\bar{Z}_\infty^{(k)}]$ would explode as $k \to \infty$.

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