EGMO and UCD Enrichment Programme in Mathematics

Selection Test 8 February 2020: Solutions

1. A, B and C are three points on a circle $\mathcal{C}$. Let BD be the bisector of angle $\angle ABC$ and $DE||AB$, where D, E lie also on $\mathcal{C}$. Prove that $|BC| = |DE|$.

   Solution. $\angle ABD = \angle DBC$ implies $AD = DC$. Also,

   $$DE||AB \Longrightarrow \angle ABD = \angle BDE$$

   so $AD = BE$. Hence $AD = DC = BE$.

   Let M be the intersection of DE and BC. Then

   $$\angle DBC = \angle BDE \Longrightarrow \Delta MBD \text{ is isosceles with } MB = MD.$$

   Similarly,

   $$\angle BCE = \angle CED \Longrightarrow \Delta MCE \text{ is isosceles with } MC = ME.$$

   This implies $BC = BM + MC = DM + ME = DE$.

   A variation: $DE||AB \Longrightarrow \angle ABD = \angle BDE$ and so $\angle BDE = \angle CBD$. Thus arc BE =arc CD. Hence

   $$\text{arc } BC = \text{arc } BE + \text{arc } CE = \text{arc } CD + \text{arc } CE = \text{arc } DE.$$

   It follows that the chords BC and DE are equal.

   Another Variation $DE||AB \Longrightarrow \angle ABD = \angle BDE$ and so $\angle BDE = \angle CBD$. On the other hand $\angle DCB = \angle DEB$ (standing on the same arc DB. So the triangles DBC and BDE are congruent since two angles and a corrsonding side agree). This implies $BC = DE$. ■

2. Let a, b, c be integers such that $a - c$ is even and $b - c$ is divisible by 3. Show that

   $$\frac{an^2}{2} + \frac{bn^3}{3} + \frac{cn}{6}$$

   is an integer for every integer n.

   Solution 1. Note that for any integer n, $n^2 - n = n(n-1)$ is always divisible by 2 (since either $n-1$ or n is). So $(n^2 - n)/2$ is always an integer and hence $n^2/2 = n/2 + m$ for some integer m.

   Similarly, $n^3 - n = (n-1)n(n+1)$ is always divisible by 3. Thus $n^3/3 = n/3 + r$ for some integer r.

   It follows that for any integer n

   $$\frac{an^2}{2} + \frac{bn^3}{3} + \frac{cn}{6} = \frac{an}{2} + \frac{bn}{3} + \frac{cn}{6} + \text{ an integer.}$$

   So it is enough to show that $\frac{an}{2} + \frac{bn}{3} + \frac{cn}{6} = (\frac{a}{2} + \frac{b}{3} + \frac{c}{6})n$ is an integer for any n.

   We must show that

   $$\frac{a}{2} + \frac{b}{3} + \frac{c}{6} = \frac{3a + 2b + c}{6}$$

   is an integer; ie. we must show that $6|3a + 2b + c$. To do this it is enough to show that $2|3a + 2b + c$ and $3|3a + 2b + c$:

   $3a + 2b + c = 4a + 2b + (c - a)$ is divisible by 2 since $c - a$ is even.

   $3a + 2b + c = 3a + 3b + (c - b)$ is divisible by 3 since $c - b$ is.

   Variation: Let $F(n) = \frac{an^2}{2} + \frac{bn^3}{3} + \frac{cn}{6}$. Then one can show that $F(n+1) = F(n)$+integer$+\frac{a}{2} + \frac{b}{3} + \frac{c}{6}$. So show this last term is an integer and conclude by induction. ■

Solution 2. We have

$$\begin{aligned}\frac{an^2}{2}+\frac{bn^3}{3}+\frac{cn}{6} &= \frac{(a-c)n^2}{2}+\frac{(b-c)n^3}{3}+c\left(\frac{n^2}{2}+\frac{n^3}{3}+\frac{n}{6}\right)\\ &= \text{integer } +c\left(\frac{n^2}{2}+\frac{n^3}{3}+\frac{n}{6}\right).\end{aligned}$$

So it is enough to show that

$$\frac{n^2}{2}+\frac{n^3}{3}+\frac{n}{6}=\frac{2n^3+3n^2+n}{6}$$

is always an integer. But $2n^3+3n^2+n=n(n+1)(2n+1)$ is divisible by 2 and by 3. ■

3. Let $ABCD$ be a convex quadrilateral. Let M, N, O, P be the midpoints of AB, BC, CD and DA respectively.

   (a) Show that $MNOP$ is a parallelogram.

   (b) Show that the area $[MNOP]$ is $1/2$ of the area $[ABCD]$.

Solution.

(a) Consider the diagonals AC and BD. MN is parallel to AC since M is the midpoint of AB and N is the midpoint of BC. Similarly, OP is parallel to AC since) is the midpoint of CD and P is the midpoint of DA. So MN is parallel to OP.

Likewise, NO and PM are both parallel to BD and so are parallel to each other.

(b)

Method 1 Let G be the point of intersection of the diagonals AC and BD.
Also let A' be the point of intersection of AG and PM. Let B' be the point of intersection of BG and MN. Let C' be the point of intersection of CG and NO. Let D' be the point of intersection of DG and OP.
$B'MA'G$ is a parallelogram by (a). Hence $GA'=MB'$ and $A'M=B'G$ $(*)$.
Moreover the triangles AMA' and MBB' are equal (since $AM=MB$, $\angle A'AM=\angle MBB'$). This implies that $[AMA']=[MBB']$ $(=x$, say), and that $A'M=B'B$, $AA'=MB'$ $(**)$.
From $(*)$ and $(**)$ we deduce that $AA'=GA'$ and $B'G=B'B$. Hence $[GA'M]=[AMA']=x$ and $[GMB']=[MBB']=x$. We conclude that $[BGA]=4x=2[B'MA'G]$.
similarly, we have $[AGD]=2[A'PD'G]$, $[DGC]=2[D'OC'G]$ and $[CGB]=2[C'NB'G]$.
Thus

$$\begin{aligned}[MNOP] &= [B'MA'G]+[A'PD'G]+[D'OC'G]+[C'NB'G]\\ &= \frac{1}{2}[BGA]+\frac{1}{2}[AGD]+\frac{1}{2}[DGC]+\frac{1}{2}[CGB]\\ &= \frac{1}{2}\left([BGA]+[AGD]+[DGC]+[CGB]\right)=\frac{1}{2}[ABCD].\end{aligned}$$

Method 2 As ABCD is convex, its area $[ABCD]$ is the area of the two triangles $[ABC]+[CDA]$.
Consider the four corner triangles that remain if we cut $MNOP$ away from $ABCD$.
We have $[MBN]=[ABC]/4$ as M is the midpoint of AB and N is the midpoint of BC.
We also have $[ODP]=[CDA]/4$ as O is the midpoint of CD and P is the midpoint of AD.
Adding these together, $[MBN]+[ODP]=[ABCD]/4$.
Likewise for the other two corner triangles, $[MAP]+[NCO]=[ABCD]/4$.
Subtracting the four corner triangles from the original $[ABCD]$ we have

$$\begin{aligned}[MNOP] &= [ABCD]-[MBN]-[ODP]-[MAP]-[NCO]\\ &= (1-\frac{2}{4})[ABCD]\\ &= \frac{1}{2}[ABCD].\end{aligned}$$

■

4. Alice and Bob play the following game. 2020 coins are placed on a table. The players take turns, each removing either one or two coins in a turn. The player to remove the last coin loses.

   Bob goes first. Which of the two players has a strategy that is guaranteed to win?

   Solution. Alice has a winning strategy: In each round, if Bob removes 1 coin, Alice removes 2, and if Bob removes 2 coins, Alice removes 1. Thus, in each complete round (of Bob's turn followed by Alice's turn) a total of 3 coins are removed. Since 2020 leaves remainder 1 on division by 3, this means eventually Bob will be left with 1 coin. ■

5. Let ABC be a triangle. Let $\mathcal{C}_1$ be the circle which passes through A and B and which is tangent to BC. Let $\mathcal{C}_2$ be the circle which passes through A and C and which is tangent to BC. Let T be the point of intersection (other than A) of $\mathcal{C}_1$ and $\mathcal{C}_2$. Show that if $\angle BAT = \angle CAT$ then $|BT| = |CT|$.

   Solution. Since BC is tangent to circle $\mathcal{C}_1$, we have that $\angle BAT = \angle TBC$.

   Similarly, since BC is tangent to circle $\mathcal{C}_2$, we have that $\angle CAT = \angle TCB$. Since $\angle BAT = \angle CAT$, it follows that $\angle TBC = \angle TCB$, so triangle TBC is isosceles, with $BT = CT$.

   ■

6. Note that the integers $6, 10, 15$ have the property that any two of them have a common divisor greater than 1, but the only common divisor of all three is 1.

   (a) Find four integers with the property that any pair of them has a common divisor greater than 1, but no triple of them has a common divisor greater than 1.

   (b) Do there exist 2020 integers with the property that every collection of 1010 of them has a common divisor greater than 1, but no collection of 1011 of them has a common divisor greater than 1?

   Solution. Let $1 \leq k < N$ be integers. We show that there is always a collection of N positive integers with the property that any k have them have a common divisor > 1 but no $k + 1$ do:

   For each different subset S of $\{1, 2, \ldots, N\}$ of size k choose a different prime number, $p(S)$ say (this can certainly be done since there are infinitely many distinct primes).

   For $i = 1, 2, \ldots, N$ let A_i be the product of all those primes $p(S)$ as S ranges over the k-element subsets which contain i.

   (Eg. if $N = 4$, $k = 2$, we need to choose $6 = \binom{4}{2}$ prime numbers:

   $$p(\{1,2\}) = 2, p(\{1,3\}) = 3, p(\{1,4\}) = 5, p(\{2,3\}) = 7, p(\{2,4\}) = 11, p(\{3,4\}) = 13.$$

   We then produce the 4 numbers

   $$\begin{aligned} A_1 &= p(\{1,2\})p(\{1,3\})p(\{1,4\}) = 2 \cdot 3 \cdot 5 = 30 \\ A_2 &= p(\{1,2\})p(\{2,3\})p(\{2,4\}) = 2 \cdot 7 \cdot 11 = 154 \\ A_3 &= p(\{1,3\})p(\{2,3\})p(\{3,4\}) = 3 \cdot 7 \cdot 13 = 273 \\ A_4 &= p(\{1,4\})p(\{2,4\})p(\{3,4\}) = 5 \cdot 11 \cdot 13 = 715.) \end{aligned}$$

   We show that our numbers $A_1, \ldots, A_N$ have the required properties:

   First, let S be any k-element subset of $\{1, \ldots, N\}$. Then the k numbers $\{A_i\}_{i \in S}$ all have the common divisor $p(S)$, by our construction.

   Now let T be any $(k+1)$-element subset of of $\{1, \ldots, N\}$. Suppose, for the sake of contradiction, that the $k + 1$ numbers $\{A_i\}_{i \in T}$ have a common prime divisor p. Then we must have $p = p(S)$ for some k-element subset S. Since T has $k + 1$ elements there is some index i in T but not in S. But then $p = p(S)$ does not divide A_i, giving us our contradiction. ■

7. The triple $(1, 5, 7)$ is such that the squares $(1, 25, 49)$ are in arithmetic progression. Show that there are infinitely many triples of positive integers (a, b, c) with greatest common divisor greater equal to 1 such that a^2, b^2 and c^2 are in arithmetic progression.

Solution 1. Note that $a^2 < b^2 < c^2$ are in arithmetic progression if and only if $b^2 - a^2 = c^2 - b^2$ if and only if $2b^2 = a^2 + c^2$.

Recall that there are infinitely many triples (u, v, b) of relatively prime integers satisfying $b^2 = u^2 + v^2$. (Such a triple is called a *Pythagorean triple.*)

Now let $b^2 = u^2 + v^2$ be a Pythagorean triple with integers $u < v < b$. Then

$$(v-u)^2 + (v+u)^2 = v^2 - 2uv + u^2 + v^2 + 2uv + u^2 = 2u^2 + 2v^2 = 2b^2$$

Thus, setting $(a, b, c) = (v - u, b, v + u)$ produces squares in arithmetic progression. It is clear that no two Pythagorean triples produce the same arithmetic progression, and that relative primality is preserved. Thus, the infinitude of Pythagorean triples implies there are infinitely many square arithmetic progressions. ■

Solution 2. If we let $x = c-b$ and $y = b-a$ the equations $2b^2 = a^2+c^2$ is equivalent to $x^2+y^2 = 2b(y-x)$. Fixing $y - x = 2$ and letting $x = 2n$ for any n we get a solution with $b = (x^2 + y^2)/4 = n^2 + (n + 1)^2 = 2n^2 + 2n + 1$. Converting back to our original problem (using $c = x + b, a = b - y$) gives the infinite family of triples $(a_n, b_n, c_n) = (2n^2 - 1, 2n^2 + 2n + 1, 2n^2 + 4n + 1)$.

These triples are all relatively prime: If p odd divides all three, then $p|(2n^2 + 2n + 1 - (2n^2 - 1) = 2(n + 1)$ and p divide $(2n^2 + 4n + 1) - (2n^2 + 2n + 1) = 2n$. ■

8. Let $a, b, c \geq 0$ be real numbers with $a + b + c = 1$.

Show that:

$$1 \leq \sqrt{a(1+b)} + \sqrt{b(1+c)} + \sqrt{c(1+a)} \leq 2$$

Solution. We consider first the left hand inequality. As $a + b + c = 1$ we have:

$$1 + b \geq 1 \geq a$$

So

$$\sqrt{a(1+b)} \geq \sqrt{a^2} = a$$

Applying this similarly to the other square roots and adding quickly gives:

$$\sqrt{a(1+b)} + \sqrt{b(1+c)} + \sqrt{c(1+a)} \geq a + b + c = 1$$

For the right hand inequality, we can use the *Cauchy-Schwartz inequality:*

$$(x_1y_1 + x_2y_2 + x_3y_3)^2 \leq (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) \text{ for any } x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$$

with $x_1 = \sqrt{a}, x_2 = \sqrt{b}, x_3 = \sqrt{c}$ and $y_1 = \sqrt{1+b}$, $y_2 = \sqrt{1+c}$ and $y_3 = \sqrt{1+a}$.

Alternatively, we can use the AM-GM inequality as follows:

Try the symmetric case $a = b = c = 1/3$ and note that equality holds. We have an idea to apply arithmetic mean - geometric mean inequality to find an upper bound for $\sqrt{a(1+b)}$ but doing this the obvious way doesn't work: we try:

$$\sqrt{a(1+b)} \leq \frac{a+1+b}{2}$$

But adding this over the three terms gives:

$$\sqrt{a(1+b)} + \sqrt{b(1+c)} + \sqrt{c(1+a)} \leq \frac{2a+2b+2c+3}{2} = \frac{5}{2}$$

This is true, but not strong enough. We can tighten the AM-GM inequality by scaling the factors so they are equal when $a = b = 1/3$. This gives:

$$\sqrt{a(1+b)} = \sqrt{2a \times \frac{1+b}{2}} \leq \frac{2a}{2} + \frac{1+b}{4}$$

With this modification we can add the three cyclical permutations, which gives:

$$\begin{aligned}
&\sqrt{2a \times \frac{1+b}{2}} + \sqrt{2b \times \frac{1+c}{2}} + \sqrt{2c \times \frac{1+a}{2}} \\
&\leq a + \frac{1+b}{4} + b + \frac{1+c}{4} + c + \frac{1+a}{4} \\
&= a + b + c + \frac{3+a+b+c}{4} \\
&= 2
\end{aligned}$$

This is what we had to prove. ■

9. Show that there are no integers x, y satisfying $x^2 + xy - 3y^2 = 2020$.

Solution 1. Suppose that x, y are integers satisfying $x^2 + xy - 3y^2 = 2020$. We will show that this is impossible:

Considering the quadratic in x

$$x^2 + xy - (3y^2 + 2020) = 0,$$

we deduce that

$$x = \frac{-y \pm \sqrt{y^2 + 4(3y^2 + 2020)}}{2} = \frac{-y \pm \sqrt{13y^2 + 8080}}{2}.$$

This implies that $13y^2 + 8080 = m^2$ where $m = 2x + y$ is an integer.

Since 8080 leaves remainder 7 on division by 13, this implies that $m^2 = 13s + 7$ for some integer s; ie. m^2 has remainder 7 on division by 13 $(*)$.

Note however that m^2 and $(m + 13a)^2$ have the same remainder for any integer a (since these two numbers differ by a multiple of 13). So, adding or subtracting multiples of 13 as needed, we can replace m by one of $0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6$.

However, $0^2 = 0$, $(\pm1)^2 = 1$, $(\pm2)^2 = 4$, $(\pm3)^2 = 9$, $(\pm4)^2 = 16$ gives remainder 3 (on division by 13), $(\pm5)^2 = 25$ gives remainder 12, and $(\pm6)^2 = 36$ gives remainder 10: 7 does not occur as the remainder of a square, contradicting $(*)$. So no such integers x, y can exist. ■

Solution 2. Reading the equation modulo 13 it becomes $x^2 + xy - 3y^2 \equiv 5$ and hence $x^2 - 12xy - 3y^2 \equiv 5$. This gives $(x - 6y)^2 \equiv 5$. Now show that 5 is not a square modulo 13 (as above) ■

10. A pond has 2020 lily pads arranged in a circle. At time zero, two frogs (Anthony and Clare) share the same lily pad.

Every minute, Anthony jumps over 99 lily pads in an anti-clockwise direction, to land on a pad 100 removed from where the jump started. At the same time, Clare jumps over 100 lily pads in a clockwise direction, to land on a pad 101 removed from where the jump started.

What is the first time that Anthony and Clare are again within five lily pads of each other?

Solution 1: As we are concerned with the relative position of Anthony and Clare, the answer is unchanged if Anthony remains stationary and Clare jumps 201 lily pads to the right.

When does Clare next come close to Anthony? She makes one circuit of the pond after roughly $2020/201 = 10$ jumps, but in fact after those 10 jumps she has travelled 2010 pads so is 10 lily pads short of Anthony.

After the 20 steps, she is 20 lily pads short of Anthony.

And so on, each circuit her shortfall relative to Anthony goes up by 10, until after 100 jumps she is 100 short of Anthony. But then after one more jump (101 in total) she is 101 lily pads past Anthony.

And so it goes on; with each further loop Clare overshoots Anthony by 10 pads less than the previous loop; after 111 jumps, Clare overshoots Anthony by 91; after 121 Clare overshoots Anthony by 81 and so on until after 201 jumps, Clare overshoots Anthony by 1 and they are on adjacent pads. Although the question asks for the first time Clare and Anthony are five or fewer pads apart, it turns out that the first time they are in this range, they are one pad apart.

Solution 2: After t jumps, Clare is $201t$ pads clockwise from Anthony, measured modulo 2020, so we have to solve:

$$201t \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\} \mod 2020$$

We notice that $201^2 = 40401 \equiv 1 \mod 2020$ and so we can multiply both sides by 201, to give:

$$t \in \left\{ \begin{array}{llllll} -1005 & -804 & -603 & -402 & -201 & 0 \\ 201 & 402 & 603 & 804 & 1005 & \end{array} \right\} \mod 2020$$

The first $t > 0$ for which one of these arises is $t = 201$ and this solves the problem.