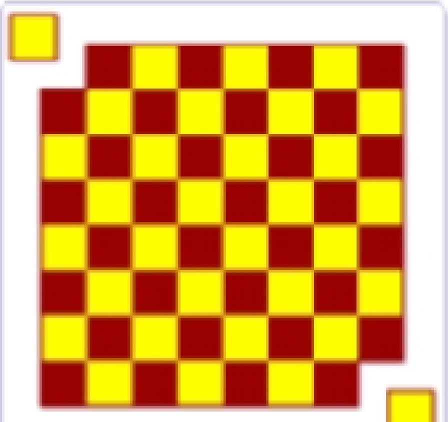
Mathematics on the Chessboard

Chess and Dominoes

Problem 1. Consider a 8×8 chessboard and remove two diametrically opposite corner unit squares. Is it possible to cover (without overlapping) the remaining 62 unit squares with dominoes?
(A domino is a 1×2 rectangle).



Solution. Two diametrically opposite corner squares that have been removed from the original chessboard have the same colour. Clearly, then, since a domino covers one yellow unit square and one red square, it is impossible to cover the remaining 62 unit squares with dominoes.

Summing 1, 0 and −1

Problem 2. In each unit square of a 8×8 array we write one of the numbers -1, 0 or 1. Is it possible that all sums on rows, columns and the two diagonals are distinct?

Summing 1, 0 and -1

Solution. No! We have $2 \cdot 8 + 2 = 18$ sums. The maximum value of such a sum is 8 and its minimum value is -8. Therefore the 18 numbers lie in the set

$$\{-8, -7, \ldots, 0, 1, \ldots 8\}.$$

Since the above set contains exactly 17 numbers, at least two of the above sums must be equal.

Products of 1 and -1 on a 7×7 board

Problem 3. (a) Is it possible to fill the unit squares of a 7×7 array with 1 or -1 such that the product of the elements in each row is 1 and the product in each column is -1?
(b) What is we consider a 8×8 board? In how many ways ?

Products of 1 and -1 on a 7×7 board

Solution. Denote by $a_1, a_2, \ldots a_7$ the product of the elements in each row and by $b_1, b_2, \ldots b_7$ product of the elements in each column.
Then $a_1 a_2 \cdots a_7$ and $b_1 b_2 \cdots b_7$ represent the product of all the elements in the array, so the must be equal. According to our condition we have

$$a_1 a_2 \cdots a_7 = 1$$

while

$$b_1 b_2 \cdots b_7 = (-1)^7 = -1$$

which is a contradiction.

Products of 1 and -1 on a 7×7 board

(b) For a 8×8 array, the above argument does not lead to any contradiction. Remark that the on column 1, the first 7 unit squares can be filled in 2^7 different ways (as each entry must be either 1 or -1) and the last unit square can be filled in only one way. Similarly, columns 2,3,4,5,6 and 7 can be filled in 2^7 ways each. The entries in first seven rows of the final column are uniquely determined.

Products of 1 and -1 on a 7×7 board

We still have to enter the number which appears in row 8,column 8. There are two cases. If there are an even number of -1's in the first seven rows and first seven columns then there will be an odd number of 1's and an even number of -1 in the first seven entries of the final column while there will be an even number of 1's and an odd number of -1 in the first seven entries of the final row. This means that -1 in the final row, final column will give the correct products. On the other hand, if there are an odd number of -1's in the first seven rows and first seven columns then there will be an even number of 1's and an odd number of -1 in the first seven entries of the final column while there will be an odd number of 1's and an even number of -1 in the first seven entries of the final row. This means that 1 in the final row, final column will give the correct products. Therefore the final answer is $(2^7)^7 \cdot 1 = 2^{14}$.

Products of 1 and -1 on a 25×25 board

Problem 4. Each unit square of a 25×25 board is filled with 1 or -1. Denote by $a_1, a_2, \ldots, a_{25}$ the products of the elements by rows and by $b_1, b_2, \ldots, b_{25}$ the product of the elements by columns. Prove that

$$a_1 + a_2 + \cdots + a_{25} + b_1 + b_2 + \cdots + b_{25} \neq 0.$$

Products of 1 and -1 on a 25×25 board

Solution.
Remark first that $a_1 a_2 \cdots a_{25}$ and also $b_1 b_2 \cdots b_{25}$ represents the product of all the numbers on the chessboard. Therefore,

$$a_1 a_2 \cdots a_{25} = b_1 b_2 \cdots b_{25}. \quad (1)$$

Denote by k (resp. p) the number of -1 between a_1, a_2, … a_{25} (resp. b_1, b_2,… b_{25}). Then (1) reads

$$(-1)^k = (-1)^p,$$

that is, k and p have the same parity. Now

$$\begin{aligned} a_1 + a_2 + \cdots + a_{25} \quad & + b_1 + \cdots + b_{25} = \\ & (25 - k) - k + (25 - p) - p \\ = \quad & 2(25 - k - p) \\ = \quad & 2[25 - (k + p)] \end{aligned}$$

which is never zero since $k + p$ is even (why?)

The Infected Chessboard

Problem 5. Seven unit cells of a 8×8 chessboard are infected. In one time unit, the cells with at least two infected neighbours (having a common side) become infected. Can the infection spread to the whole square?

The Infected Chessboard

Solution. By looking at a healthy cell with 2,3 or 4 infected neighbors, we observe that the perimeter of the infected area does not increase. Initially the perimeter of the contaminated area is at most $4 \times 7 = 28$ so it never reaches $4 \times 8 = 32$. Therefore, the infection cannot spread to the whole chessboard.

The Infected Chessboard

Figure: A normal cell (white) having two infected neighbours

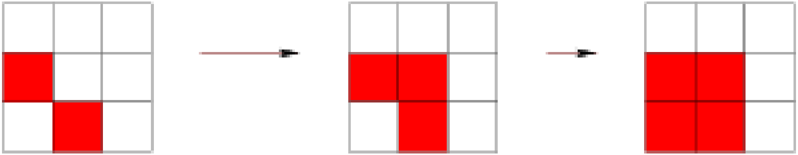


Figure: A normal cell (white) having two infected neighbours

The Infected Chessboard

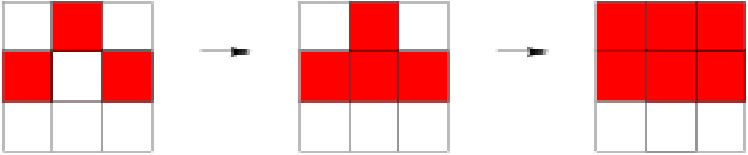


Figure: A normal cell (white) having three infected neighbours

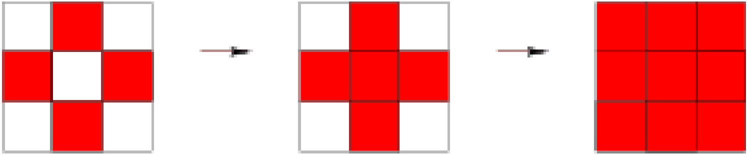


Figure: A normal cell (white) having four infected neighbours

The Infected Chessboard

Similar variant. Initially, some configuration of cells of a given $n \times n$ chessboard are infected. Then, the infection spreads as follows: a cell becomes infected if at least two of its neighbors are infected. If the entire board eventually becomes infected, prove that at least n of the cells were infected initially.

Sums of 2×2 blocks

Problem 6. The numbers $1, 2, \ldots, 81$ are randomly written in a 9×9 array. Prove that there exists a 2×2 subarray whose numbers have the sum greater than 137.

Sums of 2×2 blocks

Solution. There are exactly $8 \cdot 8 = 64$ subarrays of type 2×2.

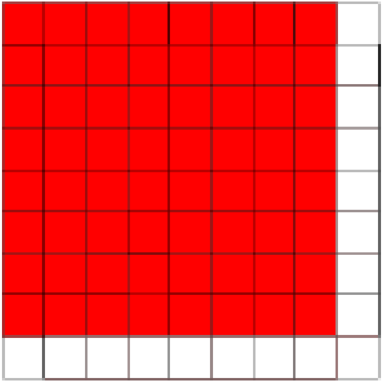


Figure: The top left unit square of any 2×2 must be one of the red squares

Sums of 2 × 2 blocks

Let

$$S_1 \leq S_2 \leq \cdots \leq S_{64}$$

be the sums of numbers written in these subarrays. Suppose that the assertion of the problem does not hold, that is, the largest of the sums in question satisfies the inequality $S_{64} \leq 137$. This also implies

$$S_1 + S_2 + \cdots + S_{64} \leq 64 \cdot 137 = 8768.$$

Sums of 2×2 blocks

On the other hand, in the above sum some of the numbers in the array are counted exactly once, some others are counted twice and some of them are counted four times.

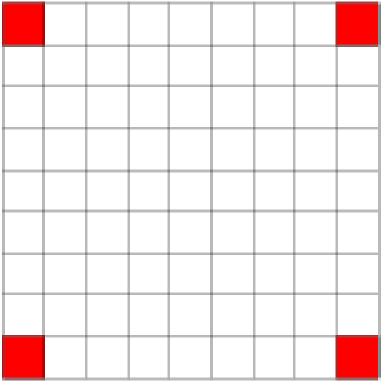


Figure: The numbers written in the red unit squares are counted only once

Sums of 2×2 blocks

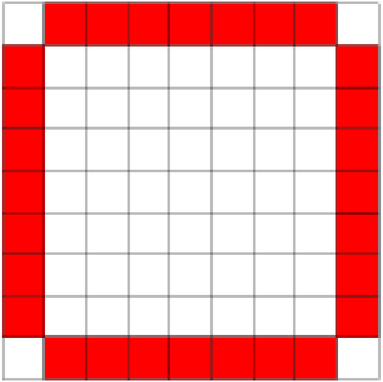


Figure: The numbers written in the red unit squares are counted exactly twice

Sums of 2×2 blocks

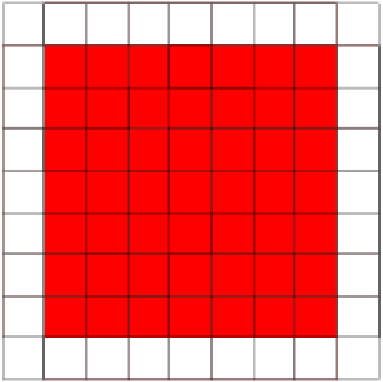


Figure: The numbers written in the red unit squares are counted exactly four times

Sums of 2 × 2 blocks

We have therefore the lower bound

$$\begin{aligned} S_1 + S_2 + \cdots + S_{64} &\geq 1(81 + 80 + 79 + 78) + 2(77 + 76 \\ &\quad + \cdots + 50) + 4(49 + 48 + \cdots + 1) \\ &= 8774, \end{aligned}$$

contradiction. Therefore, at least one of the sums in the 2×2 subarray is greater than 137.