Contents

REPORT SUMMER PROJECT
SUBHARMONIC FUNCTIONS IN REAL ANALYSIS

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Week 1

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1. Basic Facts

Definition 1.1. (Radially Symmetric Functions) We say that a function $f : \mathbb{R}^N \to \mathbb{R}$ has radial symmetry (or is radially symmetric) if $f(x) = g(|x|)$ for some $g : \mathbb{R} \to \mathbb{R}$.
(Where $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$)

Example 1.2. $f(x) = |x|^3 - 2|x| - cos|x|$ / $g(t) = t^3 - 2t - cos(t)$

Example 1.3. $f(x) = |x|^3 - 2sin(x_1) + 7ln|x|$ is not radially symmetric.

Note 1.4. If $f(x) = g(|x|)$ is radially symmetric then

$$\frac{\partial f(x)}{\partial x_i} = g'(|x|)\frac{(x_i)}{|x|}$$

And

$$\nabla f = (\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, ..., \frac{\partial f(x)}{\partial x_N}) = g'(|x|)\frac{(x_i)}{|x|} = \frac{g'(|x|)}{|x|}(x_1, x_2, ..., x_N)$$

Example 1.5. Let $f(x) = |x|^3 - 2|x| - cos|x|$. Then $\frac{\partial f(x)}{\partial x_i} = g'(|x|)\frac{x_i}{|x|} = (3|x|^2 - 2 + sin|x|)\frac{x_i}{|x|}$.

$$So \ \frac{\partial f(x)}{\partial x_i} = (3|x| - \frac{2}{|x|} + \frac{sin|x|}{|x|})x_i.$$

Theorem 1.6. *Let $f(x) = g(|x|)$ be a radially symmetric function.*

$$Then \ \frac{\partial^2 f(x)}{\partial x_i^2} = g''(|x|)\frac{x_i^2}{|x|^2} + g'(|x|)\frac{|x|^2 - x_i^2}{|x|^3}. \tag{1.1}$$

Proof. We know from Note 1.4 that $\frac{\partial f(x)}{\partial x_i} = g'(|x|)\frac{(x_i)}{|x|}$. Thus we can find $\frac{\partial^2 f(x)}{\partial x_i^2}$ by use of the chain rule. So by defintion

$$\frac{\partial^2 f(x)}{\partial x_i^2} = \frac{\partial g'(|x|)}{\partial x_i}\frac{x_i}{|x|} + g'(|x|)\frac{\partial}{\partial x_i}\frac{x_i}{|x|}$$

It then follows using the quotient rule that

$$\frac{\partial^2 f(x)}{\partial x_i^2} = g''(|x|)\frac{x_i^2}{|x|^2} + g'(|x|)(\frac{1}{|x|} - \frac{x_i^2}{|x|^3})$$

Which factors down to

$$\frac{\partial^2 f(x)}{\partial x_i^2} = g''(|x|)\frac{x_i^2}{|x|^2} + g'(|x|)\frac{|x|^2 - x_i^2}{|x|^3}$$

□

Definition 1.7. Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}$ be a C^2- function. Then

$$\Delta u = (\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_N^2}) = \sum_{i=1}^{N}(\frac{\partial^2 u}{\partial x_i^2})$$

is called the Laplace operator of u.

Note 1.8. Assume $u(x) = g(|x|)$. By Theorem 1.6, $\frac{\partial^2 u}{\partial x_i^2}$ can be easily calculated:

$$\frac{\partial^2 u}{\partial x_i^2} = g''(|x|)\frac{x_i^2}{|x|^2} + g'(|x|)\frac{|x|^2 - x_i^2}{|x|^3}$$

Denote $r = |x|$ and we get:

$$\frac{\partial^2 u}{\partial x_i^2} = g''(r)\frac{x_i^2}{r^2} + g'(r)\frac{r^2 - x_i^2}{r^3}$$

So

$$\Delta u = \sum_{i=1}^{N}\frac{\partial^2 u}{\partial x_i^2} = g''(r)\frac{r^2}{r^2} + g'(r)\frac{Nr^2 - r^2}{r^3}$$

That is,

$$\Delta u = g''(r) + \frac{N-1}{r}g'(r)$$

Example 1.9. Let $u = |x|^3 - 2|x| - cos|x|$. Find $\Delta u =$?
Solution. Note that $u(x) = g(|x|)$ (u is radially symmetric) where $g(r) = r^3 - 2r - \cos r$, $g'(r)3r^2 - 2 + \sin r$, $g''(r) = 6r + \cos r$.

Hence,

$$\Delta u = g''(r) + \frac{N-1}{r}g'(r) = 6r + \cos r + (N-1)(3r - \frac{2}{r} + \frac{\sin r}{r})$$

Example 1.10. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be defined by $u(x) = \begin{cases} \frac{\sin|x|}{|x|} & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$

Prove that $-\Delta u = u$.

Proof. We can see that $u(x) = g(|x|)$ is a radially symmetric function, therefore we can use the identity from Note 1.8.

$$\Delta u = g''(r) + \frac{N-1}{r}g'(r), \text{ where } r = |x|.$$

We have that,

$$g''(r) = \frac{2\sin(r)}{r^3} - \frac{2\cos(r)}{r^2} - \frac{\sin(r)}{r}, \ \ g'(r) = \frac{\cos(r)}{r} - \frac{\sin(r)}{r^2} \quad \text{for } x \neq 0$$

Therefore,

$$\Delta u = \frac{2\sin(r)}{r^3} - \frac{2\cos(r)}{r^2} - \frac{\sin(r)}{r} + (\frac{3-1}{r})(\frac{\cos(r)}{r} - \frac{\sin(r)}{r^2})$$
$$= -\frac{\sin(r)}{r} = -u \quad \text{for } x \neq 0$$

□

Definition 1.11. (Open and closed sets in $\mathbb{R}^N$) A set $A \subset \mathbb{R}^N$ is said to be open if for any $x \in A$ there exists $r > 0$ such that $B_r(x) \subset A$.
A set $B \subset \mathbb{R}^N$ is said to be closed if $\mathbb{R}^N \backslash B$ is open.

Example 1.12. $B_r(x)$ is open in $\mathbb{R}^N$, $[-1, 1] \subset \mathbb{R}$ is closed.
$B = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, \frac{1}{n+1}, ...\} \subset \mathbb{R}$ is closed.

Note 1.13. (Characterisation of closed sets) $B \subset \mathbb{R}^N$ is closed if for any convergent sequence $(x_n) \subset B$ we also have $lim_{n\to\infty} x_n \in B$.

Definition 1.14. Connected sets in $\mathbb{R}^N$ A set $\Omega \subset \mathbb{R}^N$ is connected if the only subset $A \subset \Omega$ which is both open and closed in Ω is either $A = \phi$ or $A = \Omega$.
A domain $\Omega \subset \mathbb{R}^N$ is an open and connected set.

Example 1.15. The intervals $[a, b], (a, b), [a, b), (a, b]$ are the only connected sets on the real line.

2. Geometric Analysis

Definition 2.1. (Unit Normal Vector) We say that an open set $\Omega \subset \mathbb{R}^N$ is of class C^k if for every $x_o \in \partial\Omega$ there exists $r > 0$ such that $B(x_o, r) \cap \partial\Omega$ is the graph of a C^k function.
If Ω is of class C^1 and $x_o \in \partial\Omega$ we can define the exterior unit normal $\nu = \nu(x_o)$ and for a C^1- function $u : \overline{\Omega} \to \mathbb{R}$ we can compute the normal derivative.

$$\frac{\partial u}{\partial \nu}(x_o) = \lim_{t\to 0} \frac{u(x_o + t\nu) - u(x_o)}{t} \text{ (for } t < 0) = \nabla u(x_o) \cdot \nu$$

Example 2.2. The ball $B(x_o, r) = B_r(x_o) = \{x \in \mathbb{R}^N : |x| < r\}$ is of class C^∞.
If $x_o \in \partial B_r(0)$, the exterior unit normal is given by:

$$\nu = \nu(x_o) = \frac{x_o}{|x_o|} = \frac{x_o}{r}$$

More generally, for a ball $B_r(z)$ we have $\nu(x_o) = \frac{x_o - z}{|x_o - z|}$.

Theorem 2.3. (Divergence Theorem) *If $F = (F_1, F_2, ..., F_N) : \overline{\Omega} \to \mathbb{R}^N$ is of class C^1 then,*

$$\int_\Omega \text{div } (F) dx = \int_{\partial\Omega} F \cdot \nu d\sigma(x)$$

Theorem 2.4. (Green's Idenities) *Let $\Omega \in \mathbb{R}^N$ be an open set of class C^1.*
If $u, v \in C^2(\overline{\Omega})$ then:

$$(a) \int_\Omega v \Delta u dx = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} d\sigma - \int_\Omega \nabla u \cdot \nabla v dx$$
$$(b) \int (v\Delta u - u \Delta v) dx = \int_{\partial\Omega} (v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu}) d\sigma$$

3. Coarea formula and applications

We first recall the coarea formula.

Theorem 3.1. (Coarea formula) *Let $f : B_R(0) \to \mathbb{R}$ be a continuous function. Then*

$$\int_{B_R(0)} f(x)dx = \int_0^R \left(\int_{\partial B_r(0)} f(x) d\sigma(x) \right) dr. \tag{3.1}$$

In particular, if $f(x) = g(|x|)$ is a radially symmetric function, then

$$\int_{B_R(0)} f(x)dx = \sigma_N \int_0^R r^{N-1} g(r) dr, \tag{3.2}$$

where σ_N denotes the surface area of the unit sphere in $\mathbb{R}^N$.

Example 3.2. As an application of the above result, let us find the explicit formula for σ_N. Let $g(r) = e^{-r^2}$ in (3.2). Then

$$\int_{\mathbb{R}^N} e^{-|x|^2} dx = \sigma_N \int_0^\infty r^{N-1} e^{-r^2} dr. \tag{3.3}$$

Note that

$$\int_{\mathbb{R}^N} e^{-|x|^2} dx = \int_{\mathbb{R}^N} e^{-x_1^2 - x_2^2 - \cdots - x_N^2} dx = \left(\int_{-\infty}^\infty e^{-r^2} dr \right)^N. \tag{3.4}$$

Also

$$\left(\int_{-\infty}^\infty e^{-r^2} dr \right)^2 = \int_{\mathbb{R}^2} e^{-|x|^2} dx = 2\pi \int_0^\infty r e^{-r^2} dr = \pi.$$

Using this equality in (3.4) we find

$$\int_{\mathbb{R}^N} e^{-|x|^2} dx = \pi^{N/2}. \tag{3.5}$$

To estimate the right-hand side in (3.3) we need the following definition.

Definition 3.3. The *Gamma function* is defined by $\Gamma : [0, \infty) \to \mathbb{R}$,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{3.6}$$

By a simple integration we deduce $\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.

In particular, an induction argument yields $\Gamma(n) = (n-1)!$ for all $n \geq 1$.

We next turn to the computation of the right-hand side in (3.3). With the substitution $t = r^2$ we have

$$\int_0^\infty r^{N-1} e^{-r^2} dr = \frac{1}{2} \int_0^\infty t^{\frac{N-2}{2}} e^{-t} dt = \frac{1}{2} \Gamma\Big(\frac{N}{2}\Big).$$

Finally, combining this last equality with (3.3) and (3.5) we find

$$\sigma_N = \frac{2\pi^{N/2}}{\Gamma\left(\frac{N}{2}\right)}.$$

Example 3.4. Let ω_N denote the volume of the unit ball in $\mathbb{R}^N$. Then, by (3.2) we find

$$\omega_N = \int_{B_1(0)} 1 dx = \sigma_N \int_0^1 r^{N-1} dr = \frac{\sigma_N}{N} = \frac{2\pi^{N/2}}{N\Gamma\left(\frac{N}{2}\right)}.$$

Note 3.5. We always have $\sigma_N = N\omega_N$.

4. Harmonic functions

Definition 4.1. Let $u : \Omega \to \mathbb{R}$ be a C^2 function where $\Omega \subset \mathbb{R}^N$ is an open set. We say that u is harmonic on Ω if $\Delta u = 0$ in Ω

Example 4.2. Let $u(x) = |x|^k$ (so $g(r) = r^k$). Then
$\Delta u = 0 \Leftrightarrow g''(r) + \frac{N-1}{r} g'(r) = 0 \Leftrightarrow k(k-1)r^{k-2} + k(N-1)r^{k-2} = 0$
$\Leftrightarrow k(k+N-2) = 0 \Leftrightarrow k = 0$ or $k = 2-N$
Hence $u(x) = C = const$ is obviously harmonic and $u(x) = |x|^{2-N}$ is also harmonic.

Example 4.3. $u(x) = \log|x|$, N=2
Then $g(r) = \log r$, $\Delta u = g''(r) + \frac{N-1}{r} g'(r) = -\frac{1}{r^2} + \frac{1}{r}.\frac{1}{r} = 0$
This shows that $u(x) = \log(x)$ is a harmonic function in dimension N=2.

Note 4.4. (What happens in dimension N=1)
If $u = u(x)$ is a harmonic function in dimension N=1 then, $0 = \Delta u = u''(x) \to u(x) = Ax+B$
Thus, the only harmonic functions in dimension 1 are linear functions.

Note 4.5. (Harmonic polynomials in dimension N=2)
Degree 0: all constant polynomials $u = c$ are harmonic.
Degree 1: all linear polynomials $u(x_1, x_2) = ax_1 + bx_2$ are harmonic.
Degree 2: all quadratic polynomials $u(x_1, x_2) = a(x_1^2 - x_2^2) + bx_1x_2$ are harmonic.
Degree n: the real and imaginary parts of $(x_1 + ix_2)^n$ are harmonic functions.

For instance $u(x_1, x_2) = x_1^3 - 3x_1x_2^2$ and $v(x_1, x_2) = x_2^3 - 3x_2x_1^2$ are harmonic.

Example 4.6. Let $f :\to \mathcal{C}$ be a holomorphic (complex differentiable) function.
Then $u(x_1, x_2) = Re(f(x_1, x_2)), v(x_1, x_2) = Im(f(x_1, x_2))$ are harmonic functions.

Take for instance $f(z) = e^z = e^{x_1+ix_2} = e^{x_1}(\cos x_2 + i \sin x_2)$. So,

$$\begin{matrix} u(x_1, x_2) = Re(f) = e^{x_1} \cos x_2 \\ v(x_1, x_2) = Im(f) = e^{x_1} \sin x_2 \end{matrix} \quad \text{are both harmonic.}$$

Example 4.7. Let $u(x) = \frac{x_1x_2}{|x|^2}, x \in \mathbb{R}^2 \backslash \{0\}$. Find Δu. We have to show that $\Delta u = 0$, so we have to calculate $\dfrac{\partial^2 u}{\partial x_i^2}$ for $i = \{1, 2\}$.

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{2x_2x_1(x_1^2 - 3x_2^2)}{|x|^6}, \quad \frac{\partial^2 u}{\partial x_2^2} = \frac{-2x_1x_2(3x_1^2 - x_2^2)}{|x|^6},$$

Therefore,

$$\sum_{i=1}^{2} (\frac{\partial^2 u}{\partial x_i^2}) = \frac{2x_2x_1(x_1^2 - 3x_2^2)}{|x|^6} + \frac{-2x_1x_2(3x_1^2 - x_2^2)}{|x|^6} = \frac{-4x_1^3x_2 - 4x_1x_2^3}{|x|^6}$$
$$= \frac{-4x_1x_2(x_1^2 + x_2^2)}{|x|^6} = \frac{-4x_1x_2}{|x|^4} = \Delta u$$

Example 4.8. Let $\Omega \subset \mathbb{R}^N$ be an open set. If u and u^2 are harmonic functions then u=constant.

Proof. $\Delta u = \nabla^2 u = 0$
$\nabla^2 u^2 = 2u\nabla^2 u + 2|\nabla u|^2 = 2|\nabla u|^2 = 0$. If $2|\nabla u|^2 = 0$ then $|\nabla u| = 0$ and u has to be a constant.

□

Example 4.9. Let $\Omega \subset \mathbb{R}^N$ be an open set, and $u : \Omega \to \mathbb{R}$ be a harmonic function. Then $v = x \cdot \nabla u(x)$ is also harmonic.

Proof.

$$v(x) = (x_1 \frac{\partial u}{\partial x_1}, x_2 \frac{\partial u}{\partial x_2}, ..., x_N \frac{\partial u}{\partial x_N})$$

$$\nabla v(x) = (x_1 \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial u}{\partial x_1}, x_2 \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_2}, ..., x_N \frac{\partial^2 u}{\partial x_N^2} + \frac{\partial u}{\partial x_N})$$

$$\nabla^2 v(x) = (x_1 \frac{\partial^3 u}{\partial x_1^3} + 2\frac{\partial^2 u}{\partial x_1^2} + x_2 \frac{\partial^3 u}{\partial x_2^3} + 2\frac{\partial^2 u}{\partial x_2^2} +, ... + x_N \frac{\partial^3 u}{\partial x_N^3} + 2\frac{\partial^2 u}{\partial x_N^2})$$

$$= 2\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{N} x_i \frac{\partial^3 u}{\partial x_i^3} = \sum_{i=1}^{N} x_i \frac{\partial^3 u}{\partial x_i^3} = x \cdot \nabla^3 u = x \cdot \nabla \cdot \nabla^2 u = 0$$

□

Example 4.10. (Find All harmonic functions which are radially symmetric)
Solution. $u = g(r)$, $r = |x|$, $\Delta u = 0$ in $\mathbb{R}^N$
Thus $0 = \Delta u = g''(r) + \frac{N-1}{r} g'(r)$ for all $r > 0$
This shows that

$$g''(r) + \frac{N-1}{r} g'(r) = 0$$
$$rg''(r) + N - 1g'(r) = 0$$

Multiply by r^{N-2} and get $r^{N-1}g''(r) + N - 1r^{N-2}g'(r) = 0$
that is, $[r^{N-1}g'(r)]' = 0$ for all $r > 0$.
Hence $r^{N-1}g'(r) = C \Rightarrow g'(r) = Cr^{1-N}$ for all $r > 0$.
So

$$g(r) = \begin{cases} c_1 r^{2-N} + c_2 & \text{for some} \quad c_1, c_2 \in \mathbb{R}, N \geq, N = 1 \\ c_1 lnr + c_2 & \text{for some} \quad c_1, c_2 \in \mathbb{R}, if N = 2 \end{cases}$$

(Conclusion): The only radially symmetric harmonics functions are,

$$u(x) = \begin{cases} c_1 r^{2-N} + c_2 & \text{for some} \quad c_1, c_2 \in \mathbb{R}, N \geq, N = 1 \\ c_1 lnr + c_2 & \text{for some} \quad c_1, c_2 \in \mathbb{R}, if N = 2 \end{cases}$$

Definition 4.11. (Fundamental Solutions of Laplace equation) The function,

$$E(x) = \begin{cases} \frac{1}{2\pi} ln|x|, & \text{if} \quad N = 2 \\ \frac{1}{(2-N)\sigma_N} |x|^{2-N} & \text{if} \quad N \geq 3 \text{ or } N = 1 \end{cases}$$

Is called the fundamental solution of the Laplace equation.
Note that the constraints $\frac{1}{2\pi}$(if N=2) or $\frac{1}{(2-N)\sigma_N}$(If N≥3) are chosen so that,

$$\int_{\partial B_R(0)} \frac{\partial E}{\partial \nu} d\sigma(y) = 1, \text{ for all } R > 0$$

Proof. Take $E(x) = \frac{1}{2\pi} \ln |x|$

$$\frac{\partial E}{\partial \nu} = \nabla E \cdot \nu$$
$$\frac{\partial E}{\partial x_i} = \frac{1}{2\pi} \frac{x_i}{|x|} \frac{1}{|x|} = \frac{1}{2\pi} \frac{x_i}{|x|^2} \Rightarrow$$
$$\nabla E = (\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}) = \frac{1}{2\pi} (\frac{x_1}{|x|^2}, \frac{x_2}{|x|^2}) = \frac{1}{2\pi} \frac{x}{|x|^2}$$

Note ($\nu = \dfrac{x}{|x|}$)

$$\frac{\partial E}{\partial \nu}(x) = \nabla E(x) \cdot \nu(x) = \frac{1}{2\pi}\frac{x}{|x|^2} \cdot \frac{x}{|x|} = \frac{1}{2\pi}\frac{|x|^2}{|x|^3} = \frac{1}{2\pi}\frac{1}{|x|} \Rightarrow$$

$$\begin{aligned}
\frac{1}{2\pi}\int_{\partial B_R(0)} \frac{\partial E}{\partial \nu}(y)d\sigma(y) &= \frac{1}{2\pi}\int_{\partial B_R(0)} \frac{1}{|y|}d\sigma(y) \\
&= \frac{1}{2\pi}\int_{\partial B_R(0)} \frac{1}{R}d\sigma(y) \\
&= \frac{1}{2\pi R}|\partial B_R(0)| \\
&= \frac{1}{2\pi R} \cdot 2\pi R = 1
\end{aligned}$$

Take $E(x) = \dfrac{1}{(2-N)\sigma_N}|x|^{2-N}$:

$$\nabla E = \frac{1}{(2-N)\sigma_N}(2-N)|x|^{1-N}\frac{x}{|x|} = \frac{x}{\sigma_N |x|^N}$$

$$\begin{aligned}
\nabla E \cdot \nu = \frac{x}{\sigma_N|x|^N} \cdot \frac{x}{|x|} = \frac{|x|^2}{\sigma_N|x|^{N+1}} = \frac{1}{\sigma_N}\frac{1}{|x|^{N-1}} \frac{1}{\sigma_N}\int_{\partial B_R(0)} \frac{\partial E}{\partial \nu}(y)d\sigma(y) \\
= \frac{1}{\sigma_N}\int_{\partial B_R(0)} \frac{1}{|x|^{N-1}}(y)d\sigma(y) \\
= \frac{1}{\sigma_N}\int_{\partial B_R(0)} \frac{1}{R^{N-1}}(y)d\sigma(y) \\
= \frac{1}{\sigma R^{N-1}}\int_{\partial B_R(0)} 1(y)d\sigma(y) \\
= \frac{1}{\sigma_N R^{N-1}}|\sigma_N R^{N-1}| = 1
\end{aligned}$$

□

Theorem 4.12. (Uniqueness of Solutions for Dirichlet problem) *Assume Ω is an open set of class $C^1, f \in C(\overline{\Omega}), g \in C(\partial\Omega)$. Then, there exists at most one solution $u \in C^2(\overline{\Omega})$ such that,*

$$(1) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Proof. Assume u_1, u_2 are two solutions of (1) and denote $u = u_1 - u_2$.

$$(2) \qquad \begin{cases} \Delta u = \Delta u_1 - \Delta u_2 = 0 & \text{in } \Omega \\ u = u_1 - u_2 = 0 & \text{on } \partial\Omega \end{cases}$$

Let us now mulitply by u in the first equation of (2). Then, by Green's identities,

$$0 = \int_\Omega u\Delta u dx = \int_{\partial\Omega} u\frac{\partial u}{\partial \nu} - \int_\Omega |\Delta u|^2 dx = -\int_\Omega |\Delta u|^2 dx \Rightarrow \int_\Omega |\Delta u|^2 dx = 0$$

Since $|\Delta u|^2 > 0$, this yields $|\Delta u| \equiv 0$ in $\Omega \Rightarrow u = \textit{constant}$ in $\Omega \Rightarrow u_1 = u_2$ in $\overline{\Omega}$. □

Definition 4.13. (Averages) Let $\Omega \subset \mathbb{R}^N$ be an open set and $u : \Omega \to \mathbb{R}$ be a continuous function.
(a) The solid average of u over a ball $B_r(x_o)$ is,

$$\fint_{B_r(x)} u(y)dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy = \frac{1}{\omega_N r^N}\int_{B_r(x)} u(y)dy$$

(b) The spherical average of u over $\partial B_r(x)$ is,

$$\fint_{\partial B_r(x)} u(y)d\sigma(y) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)d\sigma(y) = \frac{1}{\sigma_N r^{N-1}}\int_{\partial B_r(x)} u(y)d\sigma(y)$$

Theorem 4.14. (Mean Value Property for harmonic functions) *Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in C^2(\Omega)$ be a harmonic function and $B_r(x) \subset \Omega$. Then,*

$$u(x) = \fint_{B_r(x)} u(y)dy \quad \text{and} \quad u(x) = \fint_{\partial B_r(x)} u(y)d\sigma(y)$$

Proof. Recall Green's formula,

$$\int_\Omega v\Delta u dy = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu}d\sigma(y) - \int_\Omega \nabla u\cdot\nabla v dy$$

By taking $v \equiv 1$ we get,

$$\int_\Omega \Delta u dy = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}d\sigma(y)$$

Hence

$$\begin{aligned}\int_{B_r(x)} \Delta u dy &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}d\sigma(y)\\ &= \int_{\partial B_r(x)} \nabla u(y)\cdot\nu(y)d\sigma(y)\\ &= \int_{\partial B_r(x)} \nabla u(y)\cdot\frac{y-x}{r}d\sigma \quad (1)\end{aligned}$$

Let $\frac{y-x}{r} = z$. Then $y = x + rz$ and $d\sigma(y) = r^{N-1}d\sigma(z)$.
Thus, by (1) we get,

$$\begin{aligned}0 = \int_{B_r(x)} \Delta u dy &= \int_{\partial B_r(x)} \nabla u(y)\cdot\nu(y)d\sigma(y)\\ &= r^{N+1}\int_{\partial B_1(0)} \nabla u(x+rz)\cdot d\sigma(z)\\ &= r^{N-1}\frac{\partial}{\partial r}\Big[\int_{\partial B_1(0)} u(x+rz)d\sigma(z)\Big] \quad \text{for all } r>0.\end{aligned}$$

It follows that the function $r \mapsto \int_{\partial B_1(0)} u(x+rz)d\sigma(z) = \varphi(r)$ is constant, so

$$\varphi(r) = \varphi(0) = \int_{\partial B_1(0)} u(x)d\sigma(z) = u(x)\int_{\partial B_1(0)} 1d\sigma(z) = u(x)\sigma_N$$

.

Hence,

$$u(x) = \frac{1}{\sigma_N}\cdot\varphi(r) = \frac{1}{\sigma_N}\int_{\partial B_1(0)} u(x+rz)d\sigma(z)$$

Denote $x + rz = y \Rightarrow z = \frac{y-x}{r} \Rightarrow d\sigma(z) = \frac{1}{r^{N-1}}d\sigma(y)$ so,

$$u(x) = \frac{1}{\sigma_N}\int_{\partial B_r(0)} u(y)\frac{1}{r^{N-1}}d\sigma(y) = \frac{1}{\sigma_N r^{N-1}}\int_{\partial B_r(x)} u(y)d\sigma(y)$$
$$= \fint_{\partial B_r(x)} u(y)d\sigma(y).$$

The proof of the equality

$$u(x) = \fint_{B_r(x)} u(y)d\sigma(y)$$

goes as follows,

$$\begin{aligned}\int_{B_r(x)} u(y)dy &= \frac{1}{\omega_N r^N}\int_0^r \Big(\int_{\partial B_s(x)} u(y)d\sigma(y)\Big)ds\\ &= \frac{1}{\omega_N r^N}\int_0^r \Big(\sigma_N s^{N-1}\fint_{\partial B_s(x)} u(y)d\sigma(y)\Big)ds\\ &= \frac{\sigma_N}{\omega_N r^N}\int_0^r \Big(s^{N-1}\fint_{\partial B_s(x)} u(y)d\sigma(y)\Big)ds\\ &= \frac{\sigma_N}{\omega_N r^N}\int_0^r s^{N-1}u(x)ds\\ &= \frac{\sigma_N u(x)}{\omega_N r^N}\int_0^r s^{N-1}ds\\ &= \frac{\sigma_N u(x)}{\omega_N r^N}\frac{r^N}{N} = \frac{\sigma_N u(x)}{N\omega_N} = u(x).\end{aligned}$$

□

Theorem 4.15. (Strong Maximum principle for harmonic functions) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $u \in C^2(\Omega)\cap C(\overline{\Omega})$ be a harmonic function. If u achieves either its maximum or minimum in Ω then u is constant.*

Proof. Let $M = \max_{x\in\overline{\Omega}}(u)$ and assume there exists $x_o \in \Omega$ such that $u(x_o) = M$.
Define $A = \{x \in \Omega : u(x) = M\}$. Note that $A \neq \emptyset$ because $x_o \in A$. Also A is closed because if $(x_n) \in A$ such that (x_n) is convergent, then $u(x_n) = M$ for all $n \geq 1$ so that $\lim_{n\to\infty} x_n = x$ satisfies $\lim_{n\to\infty} u(x_n) = M \Rightarrow x \in A$. It remains to show that A is also open. Let $x \in A$. Then $u(x) = M$ and because Ω is open, there exists $r > 0$ with $B_r(x) \subset \Omega$.
By the Mean Value Property it follows that,

$$M = u(x) = \fint_{B_r(x)} u(y)dy \leq M.$$

It follows that $u \equiv M$ on $B_r(x)$ so $B_r(x) \subset A$.
This shows that A is both open and closed in $\Omega \Rightarrow A = \Omega$ because Ω is connected. Hence $\Omega = A = \{x \in \Omega : u(x) = M\} \Rightarrow u = constant$.

□

Theorem 4.16. (Weak Maximum principle) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $u \in C^2(\Omega)\cap(\overline{\Omega})$ be a harmonic function.*
Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u, \quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u,$$

that is, u achieves both maximum and minimum values over the boundary of the domain Ω.

Note 4.17. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $u \in C^2(\Omega)\cap(\overline{\Omega})$ be a harmonic function such that $u = 1$ on $\partial\Omega$. We can show that u is a constant function.

Proof. By the Weak Maximum principle u achieves both its maximum and minimum on $\partial\Omega$'. Since u=1 on $\partial\Omega$ then the maximum and minimum of u has to be 1 on Ω. Thus $max(u) = min(u), \Rightarrow$ u is a constant function.

□

5. A Liouville Theorem

Theorem 5.1. *Let $u : \mathbb{R}^N \to \mathbb{R}, N \geq 1$ be a harmonic function. If u is bounded either from below or above then $u =$ constant.*

Proof. Replacing u by$-u$ we may assume u is bounded below ($u \geq m$). Now replacing u by $u + m$ we may assume $u \geq 0$. Let $x \in \mathbb{R}^N \backslash \{0\}$ and $R \geq |x|$. We want to show $u(x) = u(0)$. By the Mean Value Theorem,

$$u(x) = \fint_{B_R(x)} u(y)dy \quad \text{and} \quad u(0) = \fint_{B_R(0)} u(y)dy$$

So,

$$\begin{aligned} |u(x) - u(0)| &= \frac{1}{\omega_N R^N} \Big| \int_{B_R(x)} u(y)dy - \int_{B_R(0)} u(y)dy \Big| \\ &= \frac{1}{\omega_N R^N} \Big| \int_{B_R(x) \backslash B_R(0)} u(y)dy - \int_{B_R(0) \backslash B_R(x)} u(y)dy \Big| \\ &\leq \frac{1}{\omega_N R^N} \int_{\big(B_R(x) \backslash B_R(0)\big) \cup \big(B_R(0) \backslash B_R(x)\big)} u(y)dy \quad (\star) \end{aligned}$$

<u>Claim</u>: $\big(B_R(x) \backslash B_R(0)\big) \cup \big(B_R(0) \backslash B_R(x)\big) \subset B_{R+|x|}(0) \backslash B_{R-|x|}(0)$

Proof. Let $y \in \big(B_R(x) \backslash B_R(0)\big) \cup \big(B_R(0) \backslash B_R(x)\big)$.
If $y \in \big(B_R(x) \backslash B_R(0)\big)$ then $|y - x| > R$ and $|y| > R > R - |x|$.
Thus $y \leq |y - x| + |x| < R + |x|$. So $R - |x| < |y| < R + |x|$.
If $y \in B_R(x)$ then $|y| < R$ and $|y - x| > R$. Since $|y| + |x| \geq |y - x| > R$, it follows that

$$R - |x| < |y| < R < R + |x|.$$

Hence

$$y \in B_{R+|x|}(0) \backslash B_{R-|x|}(0)$$

This proves our claim. □

We now return to the proof of $(\star)$

$$\begin{aligned} |u(x) - u(0)| &\leq \frac{1}{\omega_N R^N} \int_{\big(B_R(x) \backslash B_R(0)\big) \cup \big(B_R(0) \backslash B_R(x)\big)} u(y)dy \\ &\leq \frac{1}{\omega_N R^N} \int_{B_{R+|x|}(0) \backslash B_{R-|x|}(0)} u(y)dy \end{aligned}$$

So,

$$|u(x) - u(0)| \leq \frac{1}{\omega_N R^N} \Big[\int_{B_{R+|x|}(0)} u(y)dy - \int_{B_{R-|x|}(0)} \Big]$$

$$\begin{aligned} &= \frac{(R + |x|)^N}{R^N} \frac{1}{\omega_N (R + |x|)^N} \int_{B_{R+|x|}} u(y)dy - \frac{(R - |x|)^N}{R^N} \frac{1}{\omega_N (R - |x|)^N} \int_{B_{R-|x|}} u(y)dy \\ &= \frac{(R + |x|)^N}{R^N} \fint_{B_{R+|x|}} u(y)dy - \frac{(R - |x|)^N}{R^N} \fint_{B_{R-|x|}} u(y)dy \\ &= \frac{(R + |x|)^N - (R - |x|)^N}{R^N} u(0) \longrightarrow 0 \quad \text{as } R \longrightarrow \infty. \end{aligned}$$

This shows that $u(x) = u(0)$ for all $x \in \mathbb{R}^N \Rightarrow u = constant$. □

6. Subharmonic Functions

Definition 6.1. Let $\Omega \subset \mathbb{R}^N$ be an open set. A function $u \in C^2(\Omega)$ is called subharmonic if $-\Delta u \leq 0$ in Ω. Similarly u is called superharmonic if $-\Delta u \geq 0$ in Ω.

Example 6.2. In dimension $N = 1$ any subharmonic function u is in fact a convex function (resp. any superharmonic function is a concave function).

Example 6.3. Let $u(x) = |x|^k, x \in \mathbb{R}^N \backslash \{0\}$. Find k such that u is a subharmonic function.
Solution. u is a radially symmetric function, so using Note 1.8:

$$\Delta u = g''(r) + \frac{N-1}{r} g'(r)$$

Therefore

$$\Delta u = r^{k-2} k(k + N - 2)$$

For u to be subharmonic $-\Delta u \leq 0 \Rightarrow \Delta u \geq 0$.

$$\begin{aligned} \Delta u = r^{k-2} k(k+N-2) \geq 0 &\Rightarrow k(k+N-2) \geq 0 \\ \Rightarrow k \geq 0 \quad \text{and} \quad (k+N-2) \geq 0 \quad \text{or} \quad k \leq 0 &\quad \text{and} \quad (k+N-2) \leq 0 \\ \Rightarrow k \geq \max\{0, 2-N\} &\quad \text{or} \quad k \leq \min\{0, 2-N\} \end{aligned}$$

Example 6.4. Let $u : \Omega \to \mathbb{R}$ be a positive harmonic function. Prove that $\log(u) = \ln u$ is superharmonic.

7. Mean Value Property for Subharmonic Functions

Theorem 7.1. (Mean Value Property for Subharmonic Functions)
Let $\Omega \subset \mathbb{R}^N$ be an open set, $B_r(x) \subset\subset \Omega$. Then for any subharmonic function $u : \Omega \to \mathbb{R}$ we have,

$$u(x) \leq ⨍_{\partial B_r(x)} u(y) d\sigma y \quad \textit{and} \quad u(x) \leq ⨍_{B_r(x)} u(y) dy$$

Proof. Recall Green's formula,

$$\int_\Omega v \Delta u dy = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} d\sigma(y) - \int_\Omega \nabla u \cdot \nabla v dy$$

By taking $v \equiv 1$ we get,

$$\int_\Omega \Delta u dy = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} d\sigma(y)$$

Hence

$$\begin{aligned} \int_{B_r(x)} \Delta u dy &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu} d\sigma(y) \\ &= \int_{\partial B_r(x)} \nabla u(y) \cdot \nu(y) d\sigma(y) \\ &= \int_{\partial B_r(x)} \nabla u(y) \cdot \frac{y-x}{r} d\sigma \quad (1) \end{aligned}$$

Let $\dfrac{y-x}{r} = z$. Then $y = x + rz$ and $\underline{d\sigma(y) = r^{N-1} d\sigma(z)}$.
Thus, by (1) we get,

$$\begin{aligned} 0 \leq \int_{B_r(x)} \Delta u dy &= \int_{\partial B_r(x)} \nabla u(y) \cdot \nu(y) d\sigma(y) \\ &= r^{N+1} \int_{\partial B_1(0)} \nabla u(x+rz) \cdot d\sigma(z) \\ 0 \leq r^{N-1} \frac{\partial}{\partial r} \Big[\int_{\partial B_1(0)} & u(x+rz) d\sigma(z) \Big] \quad \text{for all } r > 0. \end{aligned}$$

It follows that the function $r \mapsto \displaystyle\int_{\partial B_1(0)} u(x+rz) d\sigma(z) = \varphi(r)$ is monotone increasing, so

$$\varphi(r) \geq \varphi(0) = \int_{\partial B_1(0)} u(x) d\sigma(z) = u(x) \int_{\partial B_1(0)} 1 d\sigma(z) = u(x) \sigma_N$$

.

Hence,

$$u(x) \leq \frac{1}{\sigma_N} \cdot \varphi(r) = \frac{1}{\sigma_N} \int_{\partial B_1(0)} u(x+rz) d\sigma(z)$$

Denote $x + rz = y \Rightarrow z = \dfrac{y-x}{r} \Rightarrow d\sigma(z) = \dfrac{1}{r^{N-1}} d\sigma(y)$ so,

$$\begin{aligned} u(x) \leq \frac{1}{\sigma_N} \int_{\partial B_r(0)} u(y) \frac{1}{r^{N-1}} d\sigma(y) &= \frac{1}{\sigma_N r^{N-1}} \int_{\partial B_r(x)} u(y) d\sigma(y) \\ &= ⨍_{\partial B_r(x)} u(y) d\sigma(y) \\ u(x) &\leq ⨍_{\partial B_r(x)} u(y) d\sigma(y). \end{aligned}$$

The proof of the equality

$$u(x) \leq ⨍_{B_r(x)} u(y) d\sigma(y)$$

goes as follows,

$$\begin{aligned} \int_{B_r(x)} u(y) dy &= \frac{1}{\omega_N r^N} \int_0^r \Big(\int_{\partial B_s(x)} u(y) d\sigma(y) \Big) ds \\ &= \frac{1}{\omega_N r^N} \int_0^r \Big(\sigma_N s^{N-1} ⨍_{\partial B_s(x)} u(y) d\sigma(y) \Big) ds \\ &= \frac{\sigma_N}{\omega_N r^N} \int_0^r \Big(s^{N-1} ⨍_{\partial B_s(x)} u(y) d\sigma(y) \Big) ds \\ &\leq \frac{\sigma_N}{\omega_N r^N} \int_0^r s^{N-1} u(x) ds \\ &\leq \frac{\sigma_N u(x)}{\omega_N r^N} \int_0^r s^{N-1} ds \\ &\leq \frac{\sigma_N u(x)}{\omega_N r^N} \frac{r^N}{N} = \frac{\sigma_N u(x)}{N \omega_N} = u(x). \end{aligned}$$

□

8. Maximum Principles for Subharmonic Functions

Theorem 8.1. (Strong Maximum Principle for Subharmonic Functions) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a subharmonic function. If there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{x \in \overline{\Omega}}$ then $u = constant$.*

Proof. Let $M = \max\limits_{x \in \overline{\Omega}}(u)$ and assume there exists $x_o \in \Omega$ such that $u(x_o) = M$.
Define $A = \{x \in \Omega : u(x) = M\}$. Note that $A \neq \emptyset$ because $x_o \in A$. Also A is closed because if $(x_n) \in A$ such that (x_n) is convergent, then $u(x_n) = M$ for all $n \geq 1$ so that $\lim\limits_{n \to \infty} x_n = x$ satisfies $\lim\limits_{n \to \infty} u(x_n) = M \Rightarrow x \in A$. It remains to show that A is also open. Let $x \in A$. Then $u(x) = M$ and because Ω is open, there exists $r > 0$ with $B_r(x) \subset \Omega$.
By the Mean Value Property it follows that,

$$M = u(x) \leq \fint_{B_r(x)} u(y)dy \leq M.$$

It follows that $u \equiv M$ on $B_r(x)$ so $B_r(x) \subset A$.
This shows that A is both open and closed in $\Omega \Rightarrow A = \Omega$ because Ω is connected. Hence $\Omega = A = \{x \in \Omega : u(x) = M\} \Rightarrow u = constant$.

□

Theorem 8.2. (Weak Maximum Principle for Subharmonic Functions) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $u \in C^2(\Omega) \cap (\overline{\Omega})$ be a subharmonic function.*
Then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Proof. This follows from the Strong Maximum Principle but there is another approach. Assume by contradiction that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{\overline{\Omega}} u > \max_{\partial\Omega} u.$$

Fix $\epsilon > 0$ small enough and define

$$v(x) = u(x) + \epsilon|x - x_0|^2, x \in \overline{\Omega}$$

We take ϵ small such that

$$\max_{\partial\Omega} v(x) \leq \max_{\partial\Omega} u + \epsilon \max_{x \in \partial\Omega} |x - x_0|^2 < v(x_0) = u(x_0)$$

This means that v achieves its maximum in $\overline{\Omega}$ at a point $\tilde{x}$ inside of Ω. Then $\frac{\partial^2 v}{\partial x_i^2}(\tilde{x}) \leq 0$ for all $1 \leq i \leq N$ so $\Delta v(\tilde{x}) \leq 0$. On the other hand

$$\Delta v(\tilde{x}) = \Delta u(\tilde{x}) + 2\epsilon N \geq 2\epsilon N > 0$$

Which is a contradiction. □