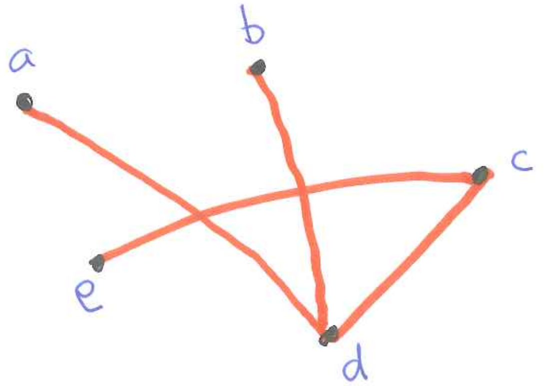
Enrichment Programme 16/2/13

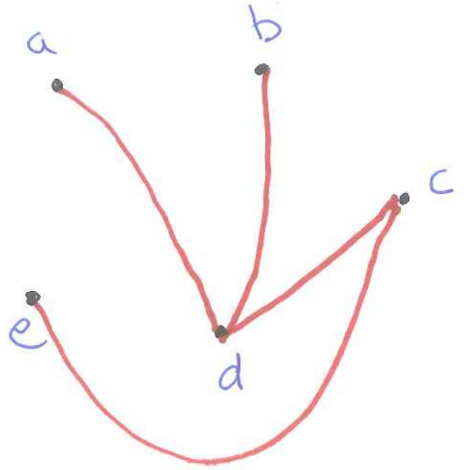
'Graphs'



5 "vertices"

4 "edges"

$\boxed{1+1+2+3+1=8}$



Same graph, represented differently.

$\deg(d) = 3 \quad \deg(a) = 1, \ \deg(c) = 2$

Observe. If $v_1, v_2, \ldots, v_n$ are the vertices of a graph then $\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2 \cdot E$

where $E = \#$ edges

From last time:

11 people exchange cards, each sending ≥ 1, and any card sent is reciprocated. Only Mary and Joe send the number. Is that number even or odd?

Graph : Vertices : people.

Edges : a joined to b if they exchange cards.

$\deg(a) = \#$ of cards sent by a

Possible degrees : $1, 2, 3, 4, \ldots, 10$.

Only one degree – n say – is repeated.

Since 11 people, all possible degrees occur.

So sum of the degrees is

$$1 + 2 + \cdots + 10 + n = 55 + n$$

This sum is even. Therefore n is odd.

<u>6</u> people at a party.

Show that

<u>either</u>

some 3 all know each other

<u>or</u>

there are 3 none of whom know each other.

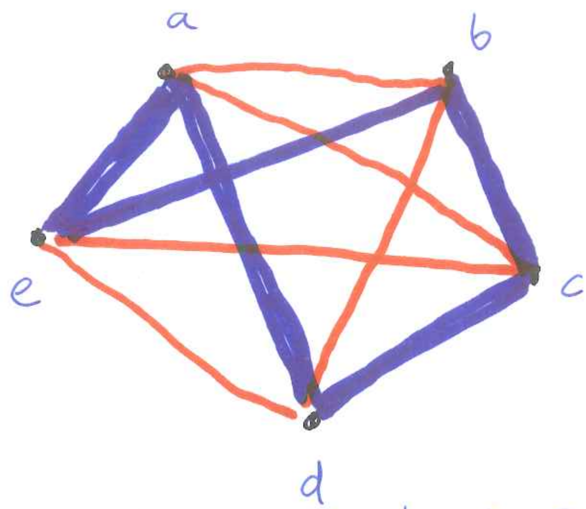
<u>Graph</u> People $\leftrightarrow$ vertices.

<u>All</u> <u>pairs</u> joined by edges (complete graph)

edges coloured red or blue according as pair know or don't know each other.

Therefore: our problem is to show that there is either a red triangle or a blue triangle in our graph.

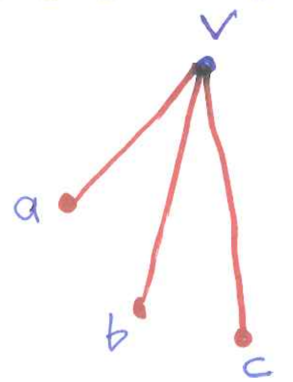
Before we start, let's try this for <u>5 people</u>



So we have shown that is possible to have $\underline{5}$ people, no three know each other, no three all strangers.

Suppose now we have 6 people. Make our graph.

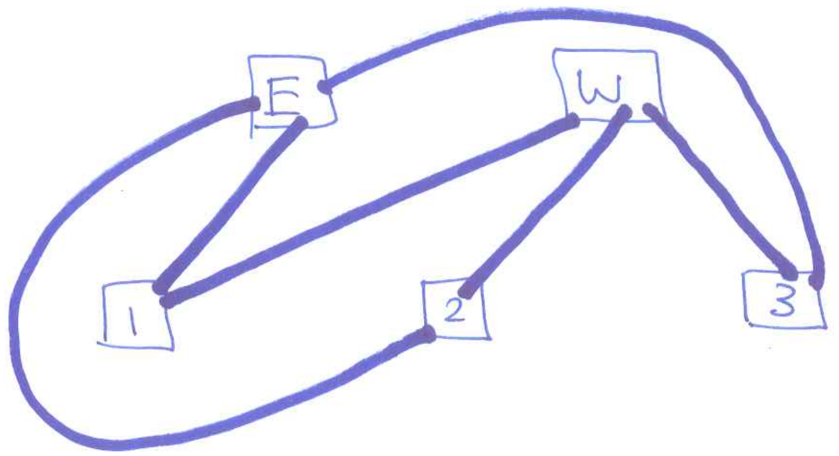
Fix a vertex v. Since there are 2 colours and 5 edges at v there must be three edges of the same colour, red say.



So there are red edges joining v to a, b, c (say)

If any pair of a, b, c is joined by a red edge we get a red triangle (involving v). Otherwise, all three are joined by a blue edge and a, b, c form a blue triangle

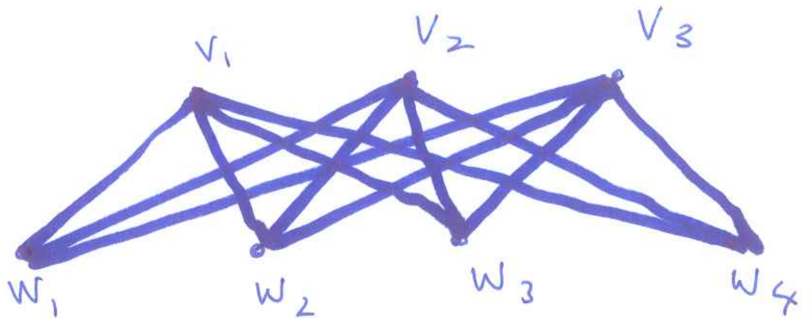
IMO (1964) 17 people correspond by mail with one another, each one with all of the rest. In their letters, 3 different topics are discussed. Each pair of correspondents deals with exactly one of these topics.
Prove that there are at least 3 people who all write to each other about the same topic.

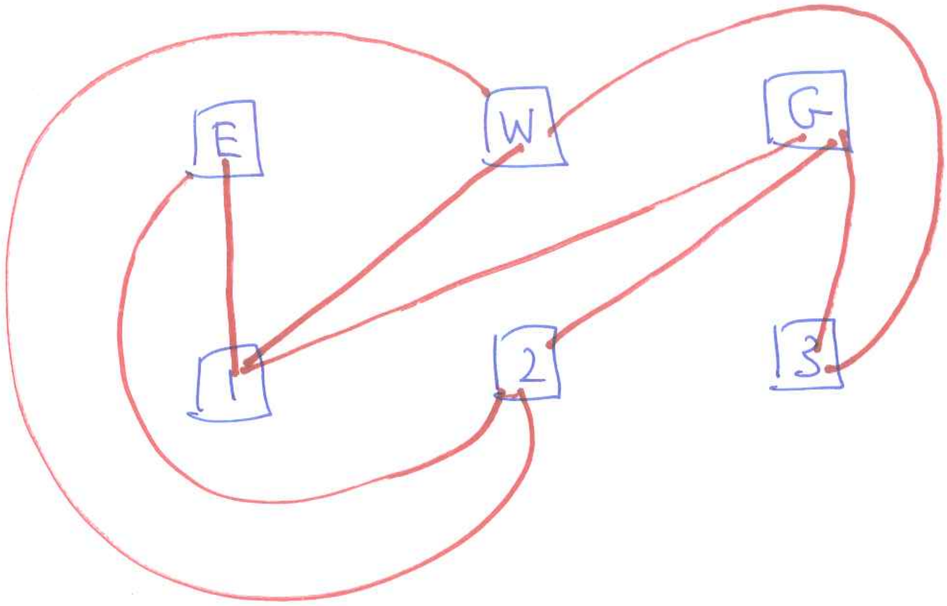


This is the bipartite graph $K_{2,3}$

The vertices partition into 2 + 3
Every one of the 2 is joined to each of the ~~three~~ 3
2 not joined to each other.
none of 3 joined to each other.

$K_{n,m}$. Eg $K_{3,4}$.



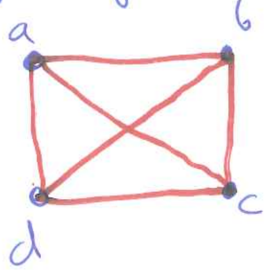


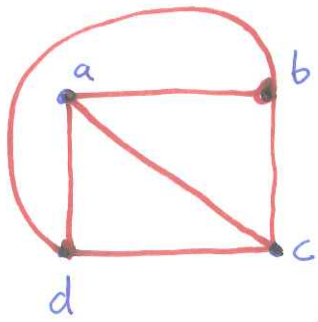
?

The question: Is $K_{3,3}$ planar?

Can the graph be drawn in a plane with no crossing edges?

K_4





$V = 4$.
$E = 6$
$R = 4$

K_4 is planar.

$\boxed{4 - 6 + 4 = 2 \checkmark}$

Let G be a connected planar graph.

It ~~divid~~ has V vertices, E edges.

It divides the plane into R regions.

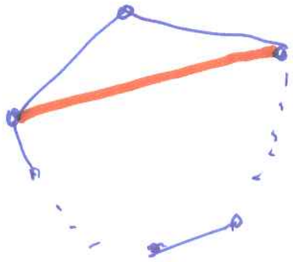
Euler's Formula It is always true that

$$V - E + R = 2.$$

Say that a connected planar graph is a maximal planar graph if adding another edge makes it non planar.

Key point In a *maximal* planar graph each region must be bounded by 3 edges.

Why?



Otherwise, we can add in 'diagonal' edges.

Theorem If G is a maximal planar graph then

$$E = 3V - 6$$

Proof $3R = 2E$ (because each region is bounded by 3 edges and each edge bounds 2 regions)

So $\underset{\text{Euler}}{6 =} 3(V - E + R) = 3V - 3E + 3R$
$= 3V - E.$

Note Any planar graph can be made into a maximal planar graph by adding more edges if needed.

Corollary If G is any (connected) planar graph

$$E \leqslant 3V - 6.$$

Exercise Deduce that if G is a connected planar graph, at least one vertex has degree $\leqslant 5$.

$K_{3,3}$ is not planar

Proof: Because it's bipartite, no triangles.

Suppose that $K_{3,3}$ were planar.

Each region is bounded by ≥ 4 edges.

$\therefore$

$$4R \leq 2E = 18 \Rightarrow \boxed{R \leq 4}$$

But $$2 = V - E + R = R - 3 \Rightarrow \boxed{R = 5}$$

A contradiction.

So $K_{3,3}$ cannot be planar after all!

Exercises Show K_5 is not planar.

Show $K_{2,n}$ is planar for any n.