MATHEMATICAL ENRICHMENT

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INEQUALITIES

BASIC INEQUALITY : $x^2 \geq 0$ x is a real number

equality holds if and only if $x=0$.
(if $x \neq 0$ then $x^2 > 0$)

It follows $x^2+y^2 \geq 0$

$x^2+y^2+z^2 \geq 0$ . . . . .

Apply this to $(x-y)^2 \geq 0$

$$x^2-2xy+y^2 \geq 0$$

$$x^2+y^2 \geq 2xy.$$

Example Show that $x^2+y^2+z^2 \geq xy+xz+yz$ where $x \geq 0$, $y \geq 0$, $z \geq 0$.

Solution:

$$x^2+y^2 \geq 2xy$$
$$x^2+z^2 \geq 2xz$$
$$y^2+z^2 \geq 2yz$$

add $2x^2+2y^2+2z^2 \geq 2xy+2xz+2yz$.

Divide by 2.

Example show $\dfrac{a^2}{a+b} \geq \dfrac{3a-b}{4}$ where $a>0, b>0$.

$$a^2+b^2 \geq 2ab.$$
$$a^2 \geq 2ab-b^2$$

add $3a^2$ $4a^2 \geq 3a^2+2ab-b^2$

$$4a^2 \geq (3a-b)(a+b).$$

$$\frac{a^2}{a+b} \geq \frac{3a-b}{4}$$

Rough work

$4a^2 \geq (3a-b)(a+b)$

$3a^2-ab+3ab-b^2$

$a^2 \geq 2ab-b^2$

$a^2+b^2 \geq 2ab$

Example Show $\dfrac{a^2}{a+b}+\dfrac{b^2}{b+c}+\dfrac{c^2}{a+c} \geq \dfrac{a+b+c}{2}$ $a>0$, $b>0$, $c>0$

Solution:
$$\frac{a^2}{a+b} \geqslant \frac{3a-b}{4}$$
$$\frac{b^2}{b+c} \geqslant \frac{3b-c}{4}$$
$$\frac{c^2}{c+a} \geqslant \frac{3c-a}{4}$$

add
$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{a+c} \geqslant \frac{3a-b}{4} + \frac{3b-c}{4} + \frac{3c-a}{4}$$
$$= \frac{2a+2b+2c}{4}$$
$$= \frac{a+b+c}{2}$$

Ir. M.O. 1999

Given $a>0, b>0, c>0, d>0, \quad a+b+c+d=1$

Show $\dfrac{a^2}{a+b} + \dfrac{b^2}{b+c} + \dfrac{c^2}{c+d} + \dfrac{d^2}{d+a} \geq \dfrac{1}{2}$.

Solution
$$\frac{a^2}{a+b} \geq \frac{3a-b}{4},$$
$$\frac{b^2}{b+c} \geqslant \frac{3b-c}{4}$$
$$\frac{c^2}{c+d} \geqslant \frac{3c-d}{4}$$
$$\frac{d^2}{d+a} \geqslant \frac{3d-a}{4}$$

add: $\quad \text{.....} \quad \geqslant \dfrac{3a-b+3b-c+3c-d+3d-a}{4}$
$$= \frac{2(a+b+c+d)}{4}$$
$$= \frac{1}{2}.$$

Consider
$$(\sqrt{x}-\sqrt{y})^2 \geqslant 0 \qquad x \geqslant 0,\ y \geqslant 0$$
$$x - 2\sqrt{x}\sqrt{y} + y \geqslant 0$$
$$\boxed{\begin{gathered} x+y \geq 2\sqrt{xy} \\ \frac{x+y}{2} \geq \sqrt{xy} \end{gathered}}$$

$$A.M. \geq G.M.$$

arithmetic mean $\geq$ geometric mean

(equality holds i.f.f. $x=y$)

<u>Example</u> Show $\frac{x}{y}+\frac{y}{x}\geq 2$ if $x>0$, $y>0$.

Solution: Apply AM–GM to $\frac{x}{y}$ and $\frac{y}{x}$

$$\frac{x}{y}+\frac{y}{x}\geq 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}=2.$$

<u>Example</u> Show $(a+b)(b+c)(a+c)\geq 8abc$ $\quad a>0,\ b>0,\ c>0$

Solution:

$a+b\geq 2\sqrt{ab}$ by AM–GM

$b+c\geq 2\sqrt{bc}$ " "

$a+c\geq 2\sqrt{ac}$ " "

Multiply
$$\begin{aligned}(a+b)(b+c)(a+c) &\geq 8\sqrt{ab}\sqrt{bc}\sqrt{ac}\\ &= 8\sqrt{a^2b^2c^2}\\ &= 8abc\end{aligned}$$

AM–GM inequality for 3 numbers

$$\boxed{\frac{x+y+z}{3}\geq\sqrt[3]{xyz}}$$

<u>Example</u> Show $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$ $\quad (a>0,\ b>0,\ c>0)$

Apply AM–GM to $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$

$$\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\geq\sqrt[3]{\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}}=1.$$

<u>Example</u> Show $x^2+\frac{4}{x}\geq 3\sqrt[3]{4}$

Apply AM–GM to $x^2, \frac{2}{x}, \frac{2}{x}$

$$\frac{x^2+\frac{2}{x}+\frac{2}{x}}{3} \geq \sqrt[3]{x^2\cdot\frac{2}{x}\cdot\frac{2}{x}} = \sqrt[3]{4}$$

multi. by 3.

AM-GM inequality for n numbers $x_1, x_2, \ldots, x_n$

$$\boxed{\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}}$$

equality if.f. $x_1=x_2=\cdots=x_n$

<u>Example</u> Show $x^6+y^6+4 \geq 6xy$ $\quad x>0$, $y>0$.

use AM-GM for 6 numbers $x^6, y^6, 1, 1, 1, 1$

get
$$\frac{x^6+y^6+1+1+1+1}{6} \geq \sqrt[6]{x^6y^6}$$
$$= xy.$$

mult. by 6

Apply AM-GM inequality to

$\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ where $a_i>0$ for all i

we get
$$\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n} \geq \sqrt[n]{\frac{1}{a_1}\cdot\frac{1}{a_2}\cdots\frac{1}{a_n}}$$
$$= \frac{1}{\sqrt[n]{a_1a_2\cdots a_n}}$$

$$\underbrace{\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}} \leq \sqrt[n]{a_1a_2\cdots a_n}$$

This is called the harmonic mean of $a_1, \ldots, a_n$

we have proved H.M. $\leq$ G.M.

So A.M. $\geq$ G.M. $\geq$ H.M.

<u>Example</u> Show $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$ when $a+b+c=1$
$a>0$
$b>0$
$c>0$

use $AM \geq HM$. This gives

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

$a+b+c=1$ $$\frac{1}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+1/c}$$

mult by 3
and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ $\Longrightarrow$ $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq 9$

Example Let $a>0, b>0, c>0$. Show that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

$\frac{6}{2}$

Add 3

$$\frac{a}{b+c} + 1 + \frac{b}{a+c} + 1 + \frac{c}{a+b} + 1 \geq \frac{3}{2} + 3$$

$$\frac{a}{b+c} + \frac{b+c}{b+c} + \frac{b}{a+c} + \frac{a+c}{a+c} + \frac{c}{a+b} + \frac{a+b}{a+b} \geq \frac{9}{2}$$

$$\frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} \geq \frac{9}{2}$$

$$(a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) \geq \frac{9}{2}$$

Mult by 2

$$(2a+2b+2c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) \geq 9$$

$$\big((a+b)+(a+c)+(b+c)\big)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) \geq 9$$

True by $\underbrace{\text{AM-HM inequality}}$.

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}$$

$$(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right) \geq n^2$$

Cauchy-Schwarz Inequality.

$$(a^2+b^2)(x^2+y^2) \geq (ax+by)^2$$

proof

$$\underbrace{(a^2+b^2)(x^2+y^2)}_{a^2x^2+a^2y^2+b^2x^2+b^2y^2} = \underbrace{(ax+by)^2}_{a^2x^2+2axby+b^2y^2} + \underbrace{(ay-bx)^2}_{a^2y^2-2aybx+b^2x^2}$$

$$\geq (ax+by)^2.$$

Cauchy-Schwarz: equality when $ay = bx$.

For 3 numbers a, b, c, and x, y, z

$$(a^2+b^2+c^2)(x^2+y^2+z^2) \geq (ax+by+cz)^2$$

Example if $x^2+y^2+z^2=1$ show $x+2y+3z \leq \sqrt{14}$.

choose $a=1$, $b=2$, $c=3$

Then $$(1^2+2^2+3^2)\underbrace{(x^2+y^2+z^2)}_{=1} \geq (x+2y+3z)^2$$

So $$x+2y+3z \leq \sqrt{14}$$

Example Given $x>0$, $y>0$, $z>0$, and $xyz=1$

show $$x+y+z \leq x^2+y^2+z^2.$$

In Cauchy-Schwarz choose $a=1$, $b=1$, $c=1$.

we get $$3(x^2+y^2+z^2) \geq (x+y+z)^2$$

By AM-GM $$\frac{x+y+z}{3} \geq \sqrt[3]{xyz} = 1$$

So $x+y+z \geq 3$.

So $$\underbrace{3(x^2+y^2+z^2)}_{\leq\ (x+y+z)^2} \leq (x+y+z)(x^2+y^2+z^2).$$

we have $$(x+y+z)^2 \leq (x+y+z)(x^2+y^2+z^2)$$

Divide by $x+y+z$.