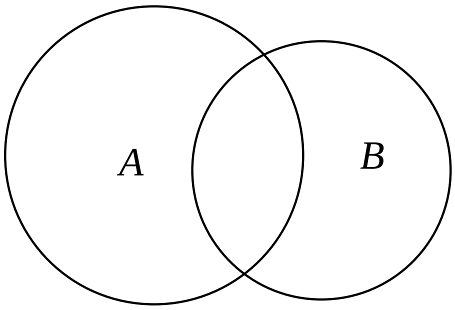
Sets and Counting

Let us remind that the **integer part** of a number x is the greatest integer that is less or equal to x. It is denoted by $[x]$.

Example $[3.1] = 3$, $[5.76] = 5$ but $[-3.1] = -4$ and $[-5.76] = -6$



Let A and B be two sets. It is easy to see (from the diagram above) that

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Problem 1. Find how many numbers from 1 to 1000 are divisible either by 7 or by 11.

Solution. Let A be the set of numbers from 1 to 1000 that are divisibly by 7 and let B be the set of numbers from 1 to 1000 that are divisible by 11. Then the set of numbers divisible either by 7 or by 11 is the set $A \cup B$.

The number of multiples of 7 from $1, 2, 3, \ldots, 100$ equals $\left[\frac{1000}{7}\right] = 142$ Hence $|A| = 142$ and similarly $|B| = \left[\frac{1000}{11}\right] = 90$. also $A \cap B =$ the set of numbers between 1 to 1000 that are divisible with both 7 and 11, so

$$|A \cap B| = \left[\frac{1000}{77}\right] = 12$$

Therefore $|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$ numbers from 1 to 1,000 are divisible eithr by 7 or by 11.

Assume now A, B and C are three sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

The above equality is usually known as the Inclusion-Exclusion Principle. Of course it can be generalized to n sets $A_1, A_2, \ldots, A_n$. The Inclusion-Exclusion Principle is simple to state and relatively easy to prove, and yet has rather spectacular applications.

Problem 2. Three sets A, B and C have the following properties: $|A| = 63$, $|B| = 91$, $|C| = 44$, $|A \cap B| = 25$, $|A \cap C| = 23$, $|C \cap B| = 21$. Also, $|A \cup B \cup C| = 139$. What is $|A \cap B \cap C|$?

Solution. We apply directly the above formula.

Problem 3. (a) How many integers between 1 and 2013 are NOT multiples of any of the numbers 2, 3 or 5?

(a) How many integers between 1 and 2013 have at least one common divisor with 6, 15 and 18?

Solution. (a) It is easier to find how many numbers in the set $\{1, 2, 3 \ldots, 2013\}$ are multiples of at leas one of the numbers 2,3 or 5. If we denote by A, B and C the set of positive integers less than 2013 which are divisible by 2 or by 3 or by 5 respectively, then $|A| = \left[\frac{2013}{2}\right] = 1006$ and similarly $|B| = \left[\frac{2013}{3}\right] = 671$, $|C| = \left[\frac{2013}{5}\right] = 402$, $|A \cap B| = \left[\frac{2013}{6}\right] = 335$, $|B \cap C| = \left[\frac{2013}{15}\right] = 134$ and $|A \cap C| = \left[\frac{2013}{10}\right] = 201$.

By the above formula we find $|A \cup B \cup C| = 1476$ and the required number is 2013-1476=537.

(b) We proceed similarly by noting that the required numbers are divisible with one of the numbers 2,3,5.

Problem 4. Let A, B and C be sets with the following properties:

- $|A| = 100$, $|B| = 50$, and $|C| = 48$.
- The number of elements that belong to exactly one of the three sets is twice the number that belong to exactly two of the sets.
- The number of elements that belong to exactly one of the three sets is three times the number that belong to all of the sets.

How many elements belong to all three sets?

Solution. Let $x_1, x_2, \ldots, x_7$ denote the number of elements in each of the regions depicted above. We have

$$\begin{aligned} x_1 + x_2 + x_4 + x_5 &= 100 \qquad (1)\\ x_2 + x_3 + x_5 + x_6 &= 50 \qquad (2)\\ x_4 + x_5 + x_6 + x_7 &= 48 \qquad (3)\\ x_1 + x_3 + x_7 &= 2(x_2 + x_4 + x_6) \qquad (4)\\ x_1 + x_3 + x_7 &= 3x_5 \qquad (5) \end{aligned}$$

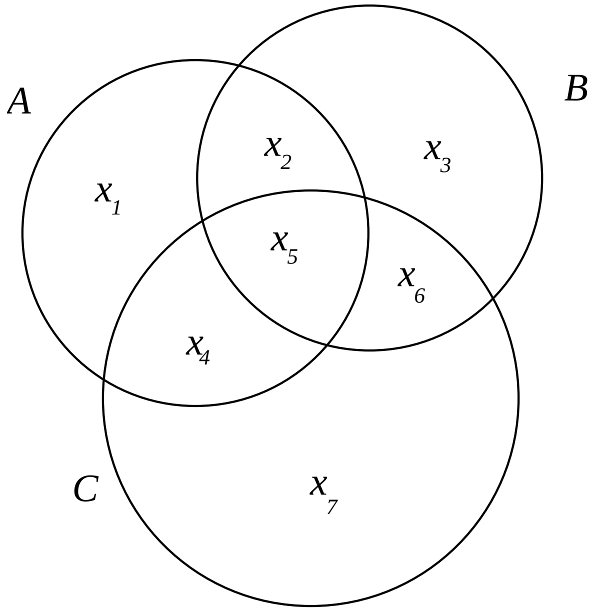
Adding the first three equations we find

$$x_1 + x_3 + x_7 + 2(x_2 + x_4 + x_6) + 3x_5 = 198.$$

Now we use eq. (4) and (5) to obtain

$$3(x_1 + x_3 + x_7) = 198$$

so $x_1 + x_3 + x_7 = 66$. From eq. (4) we find $x_2 + x_4 + x_6 = 33$ and from eq. (5) we find $x_5 = 22$. The total number of elements is 12.



Problem 5. (IMO 1966) In a math contest, three problems, A, B, and C were posed. Among the participants there were 25 who solved at least one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all participants who solved just one problem, half did not solve problem A. How many solved just problem B?

We use the same diagram as before. Thus,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 25$$
$$x_3 - x_6 - 2x_7 = 0$$
$$x_1 - x_2 - x_4 - x_5 = 1$$
$$x_1 - x_3 - x_7 = 0.$$

We next subtract equation (4) from the sum of the rst three equations above and deduce

$$x_1 + 3x_3 = 26.$$

Since $x_1 x_3 + x_7$ so $x_1 > x_3$. It follows that $4x_3 < 26$ that is, $x_3 \leq 6$.

We next analyse the cases $x_3 \in 1, 2, 3, 4, 5, 6z$ and find the only solution $x_3 = 6$ and $x_1 = 8$.

Problem 6. Set A consists of m consecutive integers whose sum is $2m$, and set B consists of $2m$ consecutive integers whose sum is m. The abslute value of the difference between the greatest element of A and the greatest element of B is 99. Find m.

Solution. Let the first element of A be $x + 1$ and that of B be $y + 1$. Then the sum of the lelements in A is $\frac{2x+m+1}{2}m = 2m$ so $2x + m = 3$

The sum of the elements in B is $\frac{2y+2m+1}{2}2m = m$ so $2y + 2m = 0$ Then we can see that $m + y = 0$

It is also given that $\mid (x + m) - (y + 2m) \mid = 99$ This gives us $\mid x \mid = 99$ Also, $2x + m = 3$, and if $x = 99$, then m will be negative. So $x = -99$ and $m = 201$.

Problem 7. Find all the finite sets A of integers with the property that for all $x \in A$ we have $x^2 - 4x + 2 \in A$.

Solution. Let $x \in A$. then $x^2-4x+2 = (x-2)^2+2 \in A$ and then $(x-2)^4+2 \in A$, $(x-2)^8+2 \in A$,..., $(x-2)^{2^n}+2 \in A$ for all $n \geq 1$. If $|x-2| > 1$ then

$$(x-2)^2+2 < (x-2)^4+2 < (x-2)^8+2 < \ldots < (x-2)^{2^n}+2 < \ldots$$

Thus, we obtain an infinite sequence of elements in A which contradicts the fact that A is finite. It follows that $|x-2| \leq 1$ which means $x \in \{1, 2, 3\}$. Remark that if $1 \in A$ then $3 \in A$ so the sets are

$$\{3\} \ \{2\}, \ \{1,3\}, \ \{2,3\}, \ \{1,2,3\}.$$

Problem 8. (Irish Math Olympiad 1997) Let p be a prime number and $A = \{1, 2, 3, \ldots, n\}$. We say that the positive integer n is $p-partitionable$ if there exists p subsets A_1, A_2, ..., A_p of A such that

(i) $A_1 \cup A_2 \cup \ldots \cup A_p = A$;

(ii) A_1, A_2, ..., A_p are disjoint (that is, $A_i \cap A_j = \emptyset$ for all $1 \leq i < j \leq p$;

(iii) The sum of elements in A_i is the same for $i = 1, 2, \ldots p$.

(a) Suppose n is $p-partitionable$. Prove that p divides n or $n+1$;

(b) Suppose that n is divisible by $2p$. Prove that n is $p-partitionable$.

Solution. (a) The sum of elements in A is $n(n+1)/2$ which must be divisible by p. thus, n or $n+1$ are divisible by p.

(b) Assume $n = 2kp$. We must find disjoint sets A_1, A_2, ..., A_p such that the sum in each A_i is $k(2kp+1)$. We arrange the elements in A in a $k \times p$ matrix as follows

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & p \\ 2p & 2p-1 & 2p-2 & \ldots & p+2 & p+1 \\ 2p+1 & 2p+2 & 2p+3 & \ldots & 3p-1 & 3p \\ 4p & 4p-1 & 4p-2 & \ldots & 3p+2 & 3p+1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2kp & \ldots & \ldots & \ldots & \ldots & (2k-1)p+1 \end{pmatrix}$$

Now, the set A_i consists of elements in column i in the above array. Hence

$$A_i\{i, 2p+1-i, 2p+i, 4p+1-i, \ldots, (2k-2)p+i, 2kp+1-i\}$$

and the sum of elements of A_i is exactly $k + 2k^2p$, so A_1, A_2, ..., A_p satisfy the conclusion.

Homework

1. Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

2. All the phone numbers in Nowheresville contain exactly 7 digits and they either start with 56, or end with 7, or both. Otherwise, the digits of the phone number can be any of the digits 09. How many possible phone numbers exist in Nowheresville?

3. (a) How many integers in the set 1, 2, 3, 4, ..., 360 have at least one prime divisor in common with 360?

   (b) Find the number of integers x such that $1 \leq x \leq 2004$ and x is relatively prime to 2005.

4. Of 28 students taking at least one subject, the number taking Math and English but not History equals the number taking Math but not History or English. No student takes English only or History only, and six students take Math and History but not English. The number taking English and History but not Math is 5 times the number taking all three subjects. If the number taking all three subjects is even and non-zero, how many are taking English and Math but not History?

5. In a survey of the chewing gum tastes of a group of baseball players, it was found that: 22 liked juicy fruit;

   25 liked spearmint;

   39 like bubble gum;

   9 like both spearmint and juicy fruit;

   17 liked juicy fruit and bubble gum;

   20 liked spearmint and bubble gum;

   6 liked all three;

   Given that four liked none of the above, how many baseball players were surveyed?

6. Mr. Brown raises chickens. Each can be described as thin or fat, brown or red, hen or rooster. Four are thin brown hens, 17 are hens, 14 are thin chickens, 4 are thin hens, 11 are thin brown chickens, 5 are brown hens, 3 are fat red roosters, 17 are thin or brown chickens. How many chickens does Mr. Brown have?

7. Consider the following information regarding three sets A, B and C. Suppose that $|A| = 14$, $|B| = 10$, $|A \cup B \cup C| = 24$ and $|A \cap B| = 6$. Consider the following assertions:

   (1) C has at most 24 members

   (2) C has at least 6 members

   (3) $A \cup B$ has exactly 18 members

   Which ones are true?