Integration:

Integration by Partial Fractions

Step 1 If you are integrating a rational function $\frac{p(x)}{q(x)}$ where degree of $p(x)$ is greater than degree of $q(x)$, divide the denominator into the numerator, then proceed to the step 2 and then 3a or 3b or 3c or 3d followed by Step 4 and Step 5.

$$\int \frac{x^2-5x+7}{x^2-5x+6}dx = \int \left(1+\frac{1}{x^2-5x+6}\right)dx = \int dx + \int \frac{dx}{x^2-5x+6}$$

Step 2 Factor denominator into irreducible factors

$$\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-3)(x-2)}$$

Step 3a If factors are linear put in form $\frac{A}{(x+c)} + \frac{B}{(x+d)}$ and find A and B.

$$\int \frac{dx}{(x-3)(x-2)} = \int \frac{A}{x-3}dx + \int \frac{B}{x-2}dx$$

Step 3b If factors are linear but squared put in form:

$$\int \frac{dx}{(x-2)(x+1)^2} = \int \frac{A}{x-2}dx + \int \frac{B}{x+1}dx + \int \frac{C}{(x+1)^2}dx$$

Step 3c If factors are quadratic put in form:

$$\int \frac{dx}{(x-1)(x^2-3x-2)} = \int \frac{A}{x-1}dx + \int \frac{Bx+C}{x^2-3x-2}dx$$

Step 3d If factors are quadratics but squared put in form:

$$\int \frac{dx}{(x-1)(x^2-3x-2)^2} = \int \frac{A}{x-1}dx + \int \frac{Bx+C}{x^2-3x-2}dx + \int \frac{Dx+E}{(x^2-3x-2)^2}dx$$

Step 4 Get common denominator on right hand side and equate coefficients:

$$\begin{aligned}\int \frac{dx}{(x-2)(x+1)^2} &= \int \tfrac{A}{x-2}dx + \int \tfrac{B}{x+1}dx + \int \tfrac{C}{(x+1)^2}dx \\ &= \int \tfrac{A(x+1)^2+B(x-2)(x+1)+C(x-2)}{(x-2)(x+1)^2}dx \\ &= \int \tfrac{(A+B)x^2+(2A-B+C)x+(A-2B-2C)}{(x-2)(x+1)^2}dx\end{aligned}$$

$$\Rightarrow \begin{cases} A+B=0 & \Rightarrow A=-B \quad \text{(equating coefficients of } x^2\text{)} \\ 2A-B+C=0 & \Rightarrow A=-C/3 \quad \text{(equating coefficients of } x\text{)} \\ A-2B-2C=1 & \Rightarrow A=1/9, B=-1/9, C=-1/3 \\ & \qquad \text{(equating numbers)} \end{cases}$$

$$\Rightarrow \int \frac{dx}{(x-2)(x+1)^2} = \int \frac{1}{9(x-2)}dx - \int \frac{1}{9(x+1)}dx - \int \frac{1}{3(x+1)^2}dx$$

Step 5 Integrate by substitution.

Let

$$\begin{aligned} u = x-2 &\quad \Rightarrow du = dx, \\ v = x+1 &\quad \Rightarrow dv = dx \end{aligned}$$

$$\begin{aligned}\Rightarrow \int \frac{dx}{(x-2)(x+1)^2} &= \frac{1}{9}\int \frac{du}{u} - \frac{1}{9}\int \frac{dv}{v} - \frac{1}{3}\int v^{-2}dv \\ &= \frac{1}{9}\ln|u| - \frac{1}{9}\ln|v| + \frac{1}{3}v + c \\ &= \frac{1}{9}\ln|x-2| - \frac{1}{9}\ln|x+1| + \frac{1}{3}(x+1) + c,\end{aligned}$$

where $c = constant$.

Examples to try

$$\begin{aligned}\int \frac{9}{(x-1)(x+2)^2}dx &= \ln\left|\frac{x-1}{x+2}\right| + \frac{3}{x+2} + c \\ \int \frac{2+3x+x^2}{x(x^2+1)}dx &= 2\ln|x| - \frac{1}{2}\ln|x^2+1| + 3\tan^{-1}x + c \\ \int \frac{x^2+2}{4x^5+4x^3+x}dx &= \ln\left|\frac{x^2}{2x^2+1}\right| + \frac{3}{4(2x^2+1)} + c\end{aligned}$$