Integration:
Integration by Parts

When to use Integration by Parts

Integration by parts is used to evaluate integrals when the usual integration techniques such as substitution and partial fractions can not be applied. The idea of integration by parts is to transform an integral which cannot be evaluated to an easier integral which can then be evaluated using one of the other integration techniques. The formula that transforms the integral we start with $\left(\int u(x)v'(x)\,dx\right)$ into the new one $\left(\int u'(x)v(x)\,dx\right)$ is given by:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

where $u(x)$ and $v(x)$ are two functions of x and $u'(x) = \frac{d}{dx}(u(x))$, $v'(x) = \frac{d}{dx}(v(x))$.

Choosing $u(x)$ and $v'(x)$

To choose which part of our integral function should be $u(x)$ and which should be $v'(x)$ we keep the following in mind:

1. We must be able to differentiate $u(x)$ to get $u'(x)$ on the RHS.
2. We must be able to integrate $v'(x)$ to get $v(x)$ on the RHS.
3. The new integral $\left(\int u'(x)v(x)\,dx\right)$ on the RHS should be easier than the original $\left(\int u(x)v'(x)\,dx\right)$ on the LHS.

Keeping the above criteria in mind a general rule of thumb is to choose $u(x)$ to be the part of the integral expression which is first in the following list which has the mnemonic LIATE:

- **L**ogarithm functions
- **I**nverse Trigonometric functions (*Examples:* $\tan^{-1}(x)$, $\cos^{-1}(x)$)
- **A**lgebraic expressions including polynomials (*Examples:* x^2, x^4, Constants)
- **T**rigonometric functions (*Examples:* $\cos(x)$, $\sin(x)$)
- **E**xponential functions

Note: $v'(x)$ should then be chosen to be the part of the expression to be integrated that remains after the choice of $u(x)$.

Basic Example: $\int xe^{2x}\,dx$

Using our selection criteria for $u(x)$ we see that that $u(x) = x$ will be our choice as algebraic expressions (x is a polynomial) comes before exponentials on the LIATE selection list.

$$u(x) = x \qquad\qquad v'(x) = e^{2x}$$

$$u'(x) = \frac{d}{dx}(x) = 1 \qquad\qquad v(x) = \int e^{2x}\,dx = \frac{e^{2x}}{2}$$

Now substituting this into the formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\int (x)(e^{2x})\,dx = (x)\left(\frac{e^{2x}}{2}\right) - \int (1)\left(\frac{e^{2x}}{2}\right)\,dx = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx$$

$$\boxed{\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) + C = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C}$$

Difficult Example: $\int \cos(2x)\, e^{x}\,dx$

Using our selection criteria above for $u(x)$ we see that that $u(x) = \cos(2x)$ will be our choice as Trigonometric functions comes before exponentials on the selection list.

$$u(x) = \cos(2x) \qquad v'(x) = e^x$$

$$u'(x) = \frac{d}{dx}(\cos(2x)) = -2\sin(2x) \qquad v(x) = \int e^x\,dx = e^x$$

Now substituting this into the formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\int (\cos(2x))(e^x)\,dx = (\cos(2x))(e^x) - \int (-2\sin(2x))\,(e^x)\,dx$$

$$\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\int \sin(2x)e^x\,dx$$

But now it appears we have a problem. The new integral ($\int \sin(2x)e^x\,dx$) does not seem any easier than the original. However if we note the change from the original integral to this new one we can see if we repeat this integration by parts on the new integral we will arrive back once again to our starting integral. We therefore employ integration by parts a second time on the new integral ($\int \sin(2x)e^x\,dx$) where our selection criteria gives us:

$$u(x) = \sin(2x) \qquad v'(x) = e^x$$

$$u'(x) = \frac{d}{dx}(\sin(2x)) = 2\cos(2x) \qquad v(x) = \int e^x\,dx = e^x$$

Now substituting this into the formula for integration by parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$$

$$\int (\sin(2x))(e^x)\,dx = (\sin(2x))(e^x) - \int (2\cos(2x))\,(e^x)\,dx = \sin(2x)e^x - 2\int \cos(2x)e^x\,dx$$

Substituting into the first integration by parts and moving all ($\int \cos(2x)e^x\,dx$) integrals to LHS gives:

$$\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\int \sin(2x)e^x\,dx$$

$$\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\left(\sin(2x)e^x - 2\int \cos(2x)e^x\,dx\right)$$

$$\int \cos(2x)e^x\,dx + 4\int \cos(2x)e^x\,dx = 5\int \cos(2x)e^x\,dx = \cos(2x)e^x + 2\sin(2x)e^x$$

$$\boxed{\int \cos(2x)e^x\,dx = \frac{1}{5}\,(\cos(2x)e^x + 2\sin(2x)e^x)}$$